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A NONLINEAR SINGULARLY PERTURBED VOLTERRA INTEGRODIFFERENTIAL EQUATION OCCURRING IN POLYMER RHEOLOGY

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ABSTRACT

We study the initial value problem for the nonlinear Volterra integro-differential equation

$$(+)\qquad \begin{cases} -\mu y'(t) = \int_{-\infty}^{t} a(t-s)F(y(t),y(s))ds & (t>0) \\ y(t) = g(t) & (-\infty < t \le 0)\,, \end{cases}$$

where $\mu > 0$ is a small parameter, a is a given real kernel, and F, g are given real functions; (+) models the elongation ratio of a homogeneous filament of a certain polyethylene which is stretched on the time interval $(-\infty, 0]$, then released and allowed to undergo elastic recovery for $t > 0$. Under assumptions which include physically interesting cases of the given functions a, F, g, we discuss qualitative properties of the solution of (+) and of the corresponding reduced problem when $\mu = 0$, and the relation between them as $\mu \to 0^+$, both for t near zero (where a boundary layer occurs) and for large t. In particular, we show that in general the filament does not recover its original length, and that the Newtonian term $-\mu y'$ in (+) has little effect on the ultimate recovery but significant effect during the early part of the recovery.

AMS (MOS) Subject Classifications: 45J05, 45M05, 45M99, 34J99, 73F99, 73G99

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A NONLINEAR SINGULARLY PERTURBED VOLTERRA INTEGRODIFFERENTIAL EQUATION OCCURRING IN POLYMER RHEOLOGY

A. S. Lodge[1,2], J. B. McLeod[1], and J. A. Nohel[1,3,4]

1. Introduction. We study the nonlinear Volterra integrodifferential equation

$$(1.1)\qquad -\mu y'(t) = \int_{-\infty}^{t} a(t-s)F(y(t),y(s))\,ds \quad (t > 0;\ ' = d/dt)$$

subject to the initial condition

$$(1.2)\qquad y(t) = g(t) \qquad (-\infty < t \leq 0)\ .$$

This initial value problem arises as a mathematical model for a process in polymer rheology which is described in Appendix A. In the specific problem discussed there μ is a positive parameter related to part of the viscosity, and the given real functions a, F, g take the forms

$$(1.3)\qquad a(t) = \sum_{k=1}^{m} a_k \exp(-t/\tau_k)$$

where a_k and τ_k are positive constants,

$$(1.4)\qquad F(y,z) = y^3/z^2 - z\ ,$$

and

$$(1.5)\qquad g(t) = \begin{cases} 1 & \text{if } -\infty < t \leq -t_0 \quad (t_0 > 0) \\ e^{\kappa(t+t_0)} & \text{if } -t_0 < t \leq 0\ , \end{cases}$$

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where κ is a positive constant. The unknown function y measures the ratio of the extended length to the original length of a homogeneous filament which is stretched in accordance with (1.2) on the time interval $(-\infty, 0]$ and then released. The equation (1.1) then describes the process of elastic recovery. Also of interest, both mathematically and physically, is the reduced equation

$$(1.6) \qquad 0 = \int_{-\infty}^{t} a(t-s)F(y(t), y(s))ds \qquad (t > 0),$$

with $y(t) = g(t)$ on $(-\infty, 0)$, and, in particular, the relation between the solutions of (1.1), (1.2) and of (1.6) as $\mu \to 0^+$; for small $\mu > 0$, (1.1), (1.2) may be regarded as a singular perturbation of (1.6).

The purpose of this paper is to discuss the qualitative behaviour of solutions of (1.1), (1.2) on the one hand and of (1.6) on the other, and the relation between them as $\mu \to 0^+$. Our analysis is not confined to the specific forms of a, F, g in (1.3) - (1.5), but rather we abstract the essential properties of these functions. (See also (A28) - (A33)). Two results of particular interest are: (i) in general, the filament does not return to its original length, as confirmed by experiments (see Appendix A and Theorems 3 and 4); (ii) similarly to behaviour in singular perturbation problems for ordinary differential equations, the solution of (1.1), (1.2) for small $\mu > 0$ decreases rapidly as μ decreases to the solution of (1.6) near $t = 0$, becoming close to it in a "boundary layer" time interval of

order $\mu|\log \mu|$, and thereafter remains close (Corollary 2.1 and Theorems 7 and 8). The introduction of the Newtonian term $-\mu y'$ in (1.1) has little effect on the ultimate behaviour of the solution.

From the mathematical point of view the theory of integrodifferential equations with decreasing convex kernels and with nonlinearities consisting of functions of one variable is well known, see e.g. [7], [18] where references to other literature are given. A part of the novelty of the present analysis is that F in (1.1) and (1.6) is a function of two variables, $y(t)$ and $y(s)$.

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2. <u>Statement of Results</u>. Let $\mathbb{R}$ denote the real numbers, $\mathbb{R}^+$ the positive real numbers, and C^k the set of k times continuously differentiable functions.

We make the following assumptions on the functions a, F, g throughout:

$$(H_a)\quad \begin{cases} a \in C^1[0,\infty);\ a(t) > 0,\ a'(t) < 0 \quad (0 \le t < \infty)\ ; \\ a \in L^1(0,\infty);\ \log a(t) \text{ is convex } \quad (0 \le t < \infty), \text{ i.e.} \\ a'(t)/a(t) \text{ is nondecreasing;} \end{cases}$$

$$(H_F)\quad \begin{cases} F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R};\ F(x,x) = 0 \quad \text{for every} \quad x > 0\ ; \\ F \in C^1(\mathbb{R}^+ \times \mathbb{R}^+) \quad \text{and} \quad F_1(y,z) > 0,\ F_2(y,z) < 0 \\ (y,z \in \mathbb{R}^+), \text{ where the subscripts denote partial differentiation;} \end{cases}$$

$$(H_g)\quad \begin{cases} g : (-\infty,0] \to \mathbb{R}^+;\ g(-\infty) = 1,\ g(0) > 1\ ; \\ g \in C(-\infty,0] \quad \text{and} \quad g \text{ is nondecreasing .} \end{cases}$$

It is readily verified that the specific functions defined in (1.3) - (1.5) satisfy (H_a), (H_F) and (H_g) respectively. While for the majority of results these are the only conditions required, some additional assumptions are needed in Theorems 3 and 4, Corollary 5.1, and Theorems 6 and 8.

The first result concerns the global existence and uniqueness of the solution of (1.1), (1.2) for a fixed $\mu > 0$ and gives some useful properties of the solution.

<u>Theorem 1</u>. <u>Let</u> (H_a), (H_F), (H_g) <u>be satisfied</u>. <u>Then for each</u> $\mu > 0$, <u>the initial value problem</u> (1.1), (1.2) <u>has a unique solution</u> $\phi(t,\mu)$ <u>on</u> $[0,\infty)$

satisfying the following properties:

(2.1) $\phi'(t,\mu) < 0$ and $1 < \phi(t,\mu) \le g(0) \quad (0 \le t < \infty)$;

(2.2) $\begin{cases} \text{if } g_1, g_2 \text{ satisfy } (H_g) \text{ and if } g_1(t) \ge g_2(t) \quad (-\infty < t \le 0), \\ \text{then the corresponding solutions } \phi_1(t,\mu), \phi_2(t,\mu) \text{ of (1.1)}, \\ \text{(1.2) satisfy } \phi_1(t,\mu) \ge \phi_2(t,\mu) \quad (0 \le t < \infty) ; \end{cases}$

(2.3) $\begin{cases} \text{if } \mu_1 > \mu_2, \text{ then the corresponding solutions of (1.1), (1.2)} \\ \text{satisfy } \phi(t,\mu_1) > \phi(t,\mu_2) \quad (0 < t < \infty) . \end{cases}$

An immediate consequence of (2.1), (2.3) is

Corollary 1.1. Let (H_a), (H_F), (H_g) be satisfied. Then for each fixed $\mu > 0$ one has

(2.4) $$\alpha(\mu) = \lim_{t\to\infty} \phi(t,\mu) \text{ exists and } \alpha(\mu) \ge 1 ;$$

moreover, if $\mu_1 > \mu_2$, then $\alpha(\mu_1) \ge \alpha(\mu_2) \ge 1$. Theorem 1 is proved in Section 3.

Remark 1.1. In the special case $a(t) \equiv Ae^{-\lambda t}$, $A > 0$, $\lambda > 0$ which satisfies (H_a) (with $\frac{a'(t)}{a(t)} \equiv -\lambda$), conclusions (2.1) and (2.4) can be strengthened respectively to the following:

(2.1') $\phi'(t,\mu) < 0$ and $1 < y_0 < \phi(t,\mu) \le g(0) \quad (0 \le t < \infty)$,

(2.4') $$\alpha(\mu) = \lim_{t\to\infty} \phi(t,\mu) \text{ exists and } \alpha(\mu) \ge y_0 > 1 ,$$

where (see the proof of (2.5) in Theorem 2) y_0 is uniquely defined by the equation

$$\int_{-\infty}^{0} e^{\lambda s} F(y_0, g(s))ds = 0 .$$

The proof of Remark 1.1 is carried out in the course of proving (2.1) (Case (ii)).

Remark 1.2. If one is interested only in global existence, rather than further properties of solutions of (1.1), (1.2), then the following existence result can readily be established.

Proposition 1. *Let* $a \in L^1(0,\infty)$, $a(t) > 0$ $(0 \leq t < \infty)$; *let* g *satisfy* (H_g); *let* $F \in C(\mathbb{R}^+ \times \mathbb{R}^+)$, $F(x,x) = 0$ *for every* $x \in \mathbb{R}^+$ *and* $F(y,z) < 0$ *for* $y < z$ *and* $F(y,z) > 0$ *for* $y > z$ $(0 < y, z < \infty)$. *Then for each fixed* $\mu > 0$ *the initial value problem* (1.1), (1.2) *has at least one solution on* $0 < t < \infty$.

The proof of this result follows from the following observations: (i) under the present hypotheses (1.1), (1.2) has a local solution on $0 \leq t < t_0$ (see e.g. [17, Lemma 1.1], also [16]); (ii) if $y(t)$ is any solution of (1.1), (1.2) on $0 \leq t < \infty$, then a slight modification of the early part of the proof of Theorem 1 shows that $1 < y(t) \leq g(0)$ $(0 \leq t < \infty)$; (iii) in view of (i), (ii), every such local solution can be extended (but not necessarily uniquely) to the interval $[0,\infty)$ (see e.g. [17, Lemma 1.2], also [16]).

Concerning the reduced problem (1.6) we prove the following global results.

Theorem 2. *Let* (H_a), (H_F), (H_g) *be satisfied. Then* (1.6) *has a unique* *(continuous) solution* ϕ_0 *on* $[0,\infty)$ *satisfying the following properties*:

(2.5) *if* $y_0 = \phi_0(0)$, *then* $1 < y_0 < g(0)$;

if $a(t) \not\equiv Ae^{-\lambda t}$, $A > 0$, $\lambda > 0$, *then*

$$(2.6) \qquad \phi_0 \in C^1[0,\infty), \ \phi_0'(t) < 0 \ \text{and} \ 1 < \phi_0(t) \leq y_0 \ (0 \leq t < \infty) ;$$

if $a(t) \equiv Ae^{-\lambda t}$, $A > 0$, $\lambda > 0$, *then* $\phi_0(t) \equiv y_0$ $(0 \leq t < \infty)$;

$$(2.7) \qquad \begin{cases} \text{if } g_1, g_2 \text{ satisfy } (H_g) \text{ and if } g_1(t) \geq g_2(t) \ (-\infty < t \leq 0), \\ \text{then the corresponding solutions } \phi_0^{(1)}, \phi_0^{(2)} \text{ of } (1.6) \\ \text{satisfy } \phi_0^{(1)}(t) \geq \phi_0^{(2)}(t) \ (0 \leq t < \infty) ; \end{cases}$$

$$(2.8) \qquad \begin{cases} \text{if } \phi(t,\mu) \text{ is the solution of } (1.1), (1.2) \text{ for a fixed } \mu > 0, \\ \text{and if } \phi_0(t) \text{ is the solution of } (1.6), \text{ then } \phi_0(t) < \phi(t,\mu) \\ (0 \leq t < \infty) . \end{cases}$$

As a consequence of (2.4), (2.6), (2.8) we have the first statement in

Corollary 2.1. *Let* (H_a), (H_F), (H_g) *be satisfied. Then*

$$(2.9) \qquad \alpha_0 = \lim_{t \to \infty} \phi_0(t) \ \text{exists and} \ 1 \leq \alpha_0 \leq \alpha(\mu) ;$$

if $a(t) \equiv Ae^{-\lambda t}$, $A > 0$, $\lambda > 0$, *then, in fact,* $\alpha_0 = y_0 > 1$. *Moreover,*

$$(2.10) \qquad \lim_{\mu \to 0^+} \alpha(\mu) = \alpha_0 .$$

Theorem 2, and (2.10) are proved in Section 4. In Theorem 8 below we establish a more precise result than (2.10) under some additional assumptions.

The next task is to establish the physically important fact that the limiting value $\alpha(\mu)$ of the solution $\phi(t,\mu)$ of (1.1), (1.2) as $t \to \infty$ satisfies $\alpha(\mu) > 1$ $(\mu > 0)$, rather than the weak form $\alpha(\mu) \geq 1$ in (2.4).

By properties (2.8), (2.9) it suffices to prove $\alpha_0 > 1$ ($\alpha_0 = \lim_{t\to\infty} \phi_0(t)$).

Theorem 3. Let (H_a), (H_F), (H_g) be satisfied. If, in addition,

$$\int_0^\infty ta(t)dt < \infty, \tag{2.11}$$

then

$$\alpha_0 > 1 . \tag{2.12}$$

We remark that in the special case $a(t) \equiv Ae^{-\lambda t}$, $A > 0$, $\lambda > 0$, there is nothing to prove since $\phi_0(t) \equiv y_0 > 1$. Theorem 3 is proved in Section 5.

Theorem 3 is best possible in the following sense.

Theorem 4. Let (H_a), (H_g) be satisfied and let

$$F(y,z) = y - z . \tag{2.13}$$

If

$$\int_0^\infty sa(s)ds = \infty, \tag{2.14}$$

then

$$\lim_{t\to\infty} \phi_0(t) = 1 , \tag{2.15}$$

when ϕ_0 is the solution of (1.6) with F satisfying (2.13).

Remark 4.1. $F(y,z) = F_0(y - z)$, where $F_0 > 0$ is a constant, is the simplest form of F being linear consistent with assumptions (H_F).

Remark 4.2. A similar proof applied to (1.1), (1.2) shows that if the hypotheses of Theorem 4 hold, then $\lim_{t\to\infty} \phi(t,\mu) = 1$, where $\phi(t,\mu)$ is the solution of (1.1), (1.2) with F satisfying (2.13). Theorem 4 is proved in Section 6.

We now turn to the study of asymptotic behaviour as $t \to \infty$ of solutions of (1.1), (1.2) and of the reduced problem (1.6). The first result is a useful estimate for $\phi'(t,\mu)$, where ϕ is the solution of (1.1), (1.2).

<u>Theorem 5</u>. <u>Let</u> (H_a), (H_F), (H_g) <u>be satisfied. Then there exists a constant</u> $K_1 > 0$ <u>(independent of</u> μ) <u>and constants</u> $\mu_0 > 0$ <u>and</u> $\tilde{K} = \tilde{K}(\mu_0) > 0$ <u>such that the solution</u> $\phi(t,\mu)$ <u>of</u> (1.1), (1.2) <u>satisfies the estimate</u>

$$(2.16)\quad 0 < -\phi'(t,\mu) \le \frac{\tilde{K}}{\mu}\exp(-K_1 t/\mu) + \tilde{K}\int_t^\infty a(s)ds \quad (0 \le t < \infty;\ 0 < \mu \le \mu_0)\ .$$

As a consequence of Theorem 5 and the logarithmic convexity of a, one obtains

<u>Corollary 5.1</u>. <u>If, in addition,</u> (2.11) <u>is satisfied, then there exist constants</u> $\mu_0 > 0$ <u>and</u> $K = K(\mu_0) > 0$ <u>such that</u>

$$(2.17)\quad 0 < \phi(t,\mu) - \alpha(\mu) \le K\int_t^\infty (s-t)a(s)ds \quad (0 \le t < \infty;\ 0 < \mu \le \mu_0)\ ,$$

where $\alpha(\mu) = \lim_{t\to\infty} \phi(t,\mu) > 1$ (<u>by</u> Theorems 1 <u>and</u> 3). <u>Moreover</u>,

$$(2.18)\ 0 < \phi(t,\mu) - \phi_0(t) \le \alpha(\mu) - \alpha_0 + K\int_t^\infty (s-t)a(s)ds \ (0 \le t < \infty;\ 0 < \mu \le \mu_0)\ ,$$

<u>where</u> ϕ_0 <u>is the solution of</u> (1.6) <u>having</u> $\alpha_0 = \lim_{t\to\infty} \phi_0(t) > 1$ (<u>by</u> Theorems 2 <u>and</u> 3).

Theorem 5 and Corollary 5.1 are proved in Section 7.

Remark 5.1. An estimate similar to (2.16) holds for the solution ϕ_0 of the reduced problem (1.6), (but, of course, without the term $\frac{K}{\mu}\exp(-K_1 t/\mu)$). This can be proved as a special case of the proof of Theorem 5 by obtaining an estimate of the form (7.6) in Section 7, where f is now independent of μ, and using it and an estimate of the form (7.4) (also now independent of μ) in (4.2) of Section 4. Consequently, (2.17) also holds with $\phi(t,\mu)$ replaced by $\phi_0(t)$ and $\alpha(\mu)$ by α_0, and with $0 \leq t < \infty$.

In connection with the last statement in Remark 5.1 it is of interest to note that this estimate for ϕ_0 can be proved independently of Theorem 5, and we state this fact as a separate result in Theorem 6. However, it should also be noted that the estimate (2.16) (for ϕ_0') described in Remark 5.1 cannot be obtained from Theorem 6.

Theorem 6. *Let* (H_a), (H_F), (H_g) *be satisfied, and let* (2.11) *hold. Then there exists a constant* $K > 0$ *such that*

$$(2.19) \qquad 0 < \phi_0(t) - \alpha_0 \leq K \int_t^\infty (s-t)a(s)ds \qquad (0 \leq t < \infty) .$$

Theorem 6 is proved in Section 8.

Remark 6.1. One can also prove the estimate (2.17) for the solution $\phi(t,\mu)$ of (1.1), (1.2) in the manner of Theorem 6, without using (2.16).

Our next task is to establish the existence of a boundary layer in a neighbourhood of $t = 0$ as $\mu \to 0^+$. For this purpose we consider the following approximation of the problem (1.1), (1.2) for small $t \geq 0$;

$$(2.20) \qquad -\mu v'(t) = \int_{-\infty}^{0} a(-s)F(v(t),g(s))ds \qquad (t > 0;\ v(0) = g(0)) .$$

It will be observed that (2.20) is not a Volterra equation, but acts rather like an ordinary differential equation. Performing the stretching transformation

$$t = \mu\tau \tag{2.21}$$

and setting $w(\tau) = v(t)$ transforms (2.20) to

$$-\frac{dw}{d\tau} = \int_{-\infty}^{0} a(-s)F(w(\tau), g(s))ds \quad (\tau > 0;\ w(0) = g(0)) . \tag{2.22}$$

Theorem 7. *Let* (H_a), (H_F), (H_g) *be satisfied. Then the initial value problem* (2.22) *has a unique solution* $w = \xi(\tau)$ *existing on* $0 \le \tau < \infty$ *and satisfying the following properties:*

$$\lim_{\tau\to\infty} \xi(\tau) = y_0 = \phi_0(0);\ 0 < \xi(\tau) - y_0 \le (g(0) - y_0)e^{-K\tau} \quad (0 \le \tau < \infty) , \tag{2.23}$$

where ϕ_0 *is the solution of* (1.6) (*see* Theorem 2) *and* K *is some positive constant.*

Moreover, if $\phi(t,\mu)$ *is the unique solution of* (1.1), (1.2) *in Theorem* 1 *and if* $\xi(t/\mu)$ *is the unique solution of* (2.20) *for* $\mu > 0$, *then for any* $t_0 > 0$ *there exists a constant* $\bar{K} > 0$ (independent of μ) *such that*

$$|\phi(t,\mu) - \xi(t/\mu)| \le \bar{K}t + O(\mu) \quad (0 \le t \le t_0;\ \mu \to 0^+) . \tag{2.24}$$

The estimate (2.24) establishes the existence of a boundary layer in a positive neighbourhood of $t = 0$. Theorem 7 is proved in Section 9.

In Corollary 2.1 we showed that $\alpha(\mu) \to \alpha_0$ as $\mu \to 0^+$, so that the solutions $\phi(t,\mu)$ of (1.1), (1.2) and $\phi_0(t)$ of (1.6) do not differ by much for small $\mu > 0$ and for large t. Our final result makes this

precise, under the additional assumptions that $ta(t) \in L^1(0,\infty)$ and $F \in C^2(\mathbb{R}^+ \times \mathbb{R}^+)$.

Theorem 8. *Let* (H_a), (H_F), (H_g) *be satisfied. In addition, assume that* $F \in C^2(\mathbb{R}^+ \times \mathbb{R}^+)$ *and that* (2.11) *holds. Then there exist constants* $K > 0$, $\mu_0 > 0$ *and a function* $\gamma \in C^1[0,\infty)$, γ *positive, bounded and non-decreasing, such that*

$$\phi_0(t) < \phi(t,\mu) < \phi_0(t) + (g(0) - \phi_0(0))\exp(-Kt/\mu) + \gamma(t)\mu|\log\mu| \tag{2.25}$$

$$(0 \le t < \infty;\ 0 < \mu \le \mu_0)\ .$$

In particular, as an immediate consequence of (2.25), *there exists a constant* $\tilde{K} > 0$ *such that*

$$0 < \phi(t,\mu) - \phi_0(t) = O(\mu|\log\mu|),(\mu \to 0^+;\ \tilde{K}\mu|\log\mu| \le t < \infty)\ . \tag{2.26}$$

Theorem 8 is proved in Section 10.

The method of proof of Theorem 8 uses the notions of upper and lower solutions of (1.1). The necessary preliminary material, which follows the lines of well known results (see e.g. [6], [19]) is collected in Appendix B. The inequality (2.25) is established by showing that the solution ϕ_0 of the reduced problem (1.6) is a lower solution of (1.1) on $0 \le t < \infty$ and by showing that

$$w(t,\mu) = \phi_0(t) + (g(0) - \phi_0(0))\exp(-Kt/\mu) + \gamma(t)\mu|\log\mu|$$

is an upper solution for suitably chosen K and γ (i.e. that $-\mu w'(t) \le \int_{-\infty}^{t} a(t-s)F(w(t),w(s))ds$ for $0 < t < \infty$, where $w(t) = g(t)$ on $-\infty < t \le 0$). Inequality (2.25) then follows from Proposition 2B, Appendix B.

The question arises whether the order $O(\mu|\log\mu|)$ in (2.26) is best possible. In the linear case it is not: for if $F(y,z) = y - z$, one can establish the inequality

$$\phi_0(t) < \phi(t,\mu) < \phi_0(t) + (g(0) - \phi_0(0))\exp(-\frac{Kt}{\mu}) + \gamma(t)\mu$$

for $0 \leq t < \infty$ and $0 < \mu \leq \mu_0$, by the method of proof of Theorem 8. In addition, one can compute $\alpha(\mu)$ and α_0 in the linear case by the method of Laplace transforms and show that $\alpha(\mu) - \alpha_0$ is precisely of the order μ. In the general case, however, we have been unable to improve the estimate (2.25).

3. Proof of Theorem 1. The classical Picard successive approximations (or the Banach fixed point theorem) applied to the integrated form of (1.1), (1.2) show that for each fixed $\mu > 0$ there is a unique local solution $\phi(t,\mu)$ existing and in C^1 on some interval $[0,T]$, $T > 0$. To show that this solution can be continued (necessarily uniquely in view of the assumptions) to the interval $[0,\infty)$, it suffices to establish the inequalities (2.1) on any interval on which the solution $\phi(t,\mu)$ exists. For then the solution satisfies a priori upper and lower bounds, independent of T, and hence can be continued to the interval $[0,\infty)$ by a standard result [17].

To establish (2.1) on any interval on which $\phi(t,\mu)$ exists we have from (1.1), (1.2)

$$-\mu\phi'(0,\mu) = \int_{-\infty}^{0} a(-s)F(g(0),g(s))ds \ . \tag{3.1}$$

The integral clearly exists since $a \in L^1(0,\infty)$ and $F(g(0),g(s))$ is bounded on $(-\infty,0]$ by (H_F), $H_g)$. From (H_a), $a(-s) > 0$ $(-\infty < s \leq 0)$ and from (H_F), (H_g), $F(g(0),g(s)) \geq 0$ $(-\infty < s \leq 0)$, with the strict inequality holding for large negative s; therefore, $\phi'(0,\mu) < 0$. Since $\phi \in C^1$, one has by continuity that $\phi'(t,\mu) < 0$ $(0 \leq t < \alpha)$, for some $\alpha > 0$. We claim first that

$$\phi(t,\mu) > 1 \qquad (0 \leq t \leq \alpha) \ . \tag{3.2}$$

Indeed, $\phi(0,\mu) = g(0) > 1$, and by continuity (3.2) holds at least on

some interval to the right of $t = 0$. Suppose $0 < t_1 \leq \alpha$ is the first point at which $\phi(t_1, \mu) = 1$, and $1 < \phi(t, \mu) < g(0)$ $(0 < t < t_1)$. From (1.1) we have

$$(3.3) \qquad -\mu\phi(t_1, \mu) = \int_{-\infty}^{0} a(t_1 - s)F(1, g(s))ds + \int_{0}^{t_1} a(t_1 - s)F(1, \phi(s, \mu))ds \, .$$

By (H_a), $a(t_1 - s) > 0$ $(-\infty < s \leq t_1)$, and by (H_F), (H_g), $F(1, g(s)) \leq 0$ on $(-\infty, 0]$, with the strict inequality holding near zero. Since ϕ is strictly decreasing on $[0, t_1]$, we also have $F(1, \phi(s, \mu)) < 0$ $(0 \leq s < t_1)$. Therefore each integral in (3.3) is negative and $\phi'(t_1, \mu) > 0$, which, in view of $\phi'(t, \mu) < 0$ $(0 \leq t < \alpha)$, is impossible; this proves (3.2).

We next claim:

$$(3.4) \qquad \phi'(t, \mu) < 0 \, ,$$

for as long as the solution exists. Indeed, suppose for contradiction that $\alpha > 0$ is the first point at which

$$(3.5) \qquad \phi'(\alpha, \mu) = 0 \quad \text{and} \quad \phi'(t, \mu) < 0 \quad (0 \leq t < \alpha) \, .$$

By the argument of the preceding paragraph we have

$$(3.6) \qquad 1 < \phi(\alpha, \mu) < g(0) \, .$$

To prove (3.4) we compute $\phi''(\alpha, \mu)$ from (1.1) and we shall obtain an obvious contradiction of (3.5) by showing that

$$(3.7) \qquad \phi''(\alpha, \mu) < 0 \, ;$$

this implies that no such $\alpha > 0$ satisfying (3.5) exists and proves (3.4). Indeed, differentiating (1.1) (justified by (H_a), (H_F), (H_g) - note that by

(H_a), $a' \in L^1(0,\infty))$ one has

$$(3.8)\qquad \begin{cases} -\mu\phi''(t,\mu) = \int_{-\infty}^{t} a'(t-s)F(\phi(t,\mu),\phi(s,\mu))ds \\ \\ \qquad + \phi'(t,\mu)\int_{-\infty}^{t} a(t-s)F_1(\phi(t,\mu),\ \phi(s,\mu))ds\ . \end{cases}$$

Putting $t = \alpha$ and using (3.5) gives

$$(3.9)\qquad -\mu\phi''(\alpha,\mu) = \int_{-\infty}^{\alpha} a'(\alpha - s)F(\phi(\alpha,\mu),\phi(s,\mu))ds\ .$$

Thus to prove (3.7) we wish to show that

$$(3.10)\qquad I(\alpha) = \int_{-\infty}^{\alpha} a'(\alpha - s)F(\phi(\alpha,\mu),\phi(s,\mu))ds > 0\ .$$

We shall need to consider two cases:

in case (i) $a(t)$ satisfies (H_a) with $a(t) \not\equiv Ae^{-\lambda t}$, $A > 0$, $\lambda > 0$ (i.e. $a'(t)/a(t) \not\equiv -\lambda$);

in case (ii) $a(t)$ satisfies (H_a) with $a(t) \equiv Ae^{-\lambda t}$, $A > 0$, $\lambda > 0$.

<u>Case (i)</u>. Define a number $-\beta$, $\beta > 0$, by the relation $\phi(\alpha,\mu) = g(-\beta)$; $-\beta$ exists in view of (3.6) and (H_g). Since g may take the constant value $\phi(\alpha,\mu)$ on some interval $J \subset (-\infty,0]$, we define $-\beta$ uniquely by taking it to be the right-hand end point in such a case. We then have

$$(3.11)\qquad I(\alpha) = \int_{-\infty}^{-\beta} a'(\alpha - s)F(g(-\beta),g(s))ds + \int_{-\beta}^{0} a'(\alpha - s)F(g(-\beta),g(s))ds$$
$$+ \int_{0}^{\alpha} a'(\alpha - s)F(\phi(\alpha,\mu),\phi(s,\mu))ds\ .$$

Since $\phi'(\alpha,\mu) = 0$ we also have from (1.1)

$$(3.12)\qquad 0 = \int_{-\infty}^{-\beta} a(\alpha - s)F(g(-\beta),g(s))ds + \int_{-\beta}^{0} a(\alpha - s)F(g(-\beta),g(s))ds$$

$$+ \int_{0}^{\alpha} a(\alpha - s)F(\phi(\alpha,\mu),\phi(s,\mu))ds .$$

We next define the function σ by the relation

$$\sigma(s) = \frac{a'(\alpha - s)}{a(\alpha - s)} \qquad (-\infty < s \leq 0) ,$$

and we observe that the log convexity of a implies that $\sigma(s)$ is negative and nonincreasing; moreover, since $a'(t)/a(t) \not\equiv -\lambda$, $\lambda > 0$, $\sigma(s)$ is strictly decreasing, at least on some interval contained in $(-\infty, 0]$. We rewrite (3.11) in the equivalent form

$$(3.13)\qquad I(\alpha) = \int_{-\infty}^{-\beta} \sigma(s)h(s)ds + \int_{-\beta}^{\alpha} \sigma(s)h(s)ds$$

where

$$h(s) = \begin{cases} a(\alpha - s)F(g(-\beta),g(s)) & (-\infty < s \leq 0) \\ a(\alpha - s)F(\phi(\alpha,\mu),\phi(s,\mu)) & (0 < s \leq \alpha) ; \end{cases}$$

we also write (3.12) in the equivalent form

$$(3.14)\qquad 0 = \int_{-\infty}^{-\beta} h(s)ds + \int_{-\beta}^{\alpha} h(s)ds .$$

From the definition of $-\beta$ and (3.6) one has $1 < g(-\beta) < g(0)$. Therefore, (H_a), (H_F), (H_g) imply that

$$(3.15)\qquad h(s) \geq 0 \qquad (-\infty < s \leq -\beta)$$

with strict inequality for large negative s, and

(3.16) $$h(s) < 0 \qquad (-\beta < s < \alpha) .$$

Combining (3.13), (3.14) yields

(3.17) $$I(\alpha) = \int_{-\infty}^{-\beta} (\sigma(s) - \sigma(-\beta))h(s)ds + \int_{-\beta}^{\alpha} (\sigma(s) - \sigma(-\beta))h(s)ds .$$

But

(3.18) $$\sigma(s) \geq \sigma(-\beta) \quad (-\infty < s \leq -\beta) ,$$

(3.19) $$\sigma(s) \leq \sigma(-\beta) \quad (-\beta < s \leq \alpha) ,$$

with strict inequalities holding either for s negative and large or for s near α. Using (3.15), (3.16) and (3.18), (3.19) in (3.17) shows that each integral in (3.17) is nonnegative and at least one of them is positive. This proves (3.10) and hence also (3.7) and (3.4). The proof of the global existence and uniqueness of the solution $\phi(t,\mu)$ of (1.1), (1.2) and of property (2.1) in case (i) is then completed by a straightforward continuation argument.

<u>Case (ii)</u>. The above balancing argument establishing (3.7), and hence (3.4), cannot be used when $a(t) \equiv Ae^{-\lambda t}$, $A > 0$, $\lambda > 0$, since for this case it is readily established from (1.1) and (3.9) that $\phi''(\alpha,\mu) = 0$ whenever $\phi'(\alpha,\mu) = 0$. Instead we proceed as follows.

Define the number y_0, $1 < y_0 < g(0)$, by the equation $\int_{-\infty}^{0} e^{\lambda s}F(y_0, g(s))ds = 0$. (For the proof of the existence of a unique y_0

with this property in the general case of $a(t)$, which also applies here, see the proof of (2.5) in Section 4.) Since as in case (i) $\phi'(0,\mu) < 0$, a repetition of the argument (3.1) - (3.3) above with $a(t) = Ae^{-\lambda t}$, $A > 0$, $\lambda > 0$, and "1" replaced by y_0 in (3.2) and the paragraph following (3.2), shows that $\phi(t,\mu) > y_0$ for $t \geq 0$, so long as $\phi'(t,\mu) < 0$. To show that $\phi'(t,\mu) < 0$ for all $t \geq 0$, differentiation of (1.1) with $a(t) = Ae^{-\lambda t}$, $A > 0$, $\lambda > 0$, yields the equation

$$\mu\phi''(t,\mu) + \phi'(t,\mu)[\lambda\mu + \int_{-\infty}^{t} e^{-\lambda(t-s)} F_1(\phi(t,\mu),\phi(s,\mu))ds] = 0 ,$$

with $\phi(0,\mu) = g(0)$, $\phi'(0,\mu) = -\frac{1}{\mu}\int_{-\infty}^{0} e^{\lambda s} F(g(0),g(s))ds < 0$. Since $F_1 > 0$, $\lambda > 0$, $\mu > 0$, an elementary argument shows that $\phi'(t,\mu) < 0$ so long as the solution $\phi(t,\mu)$ exists. This completes the proof of (2.1') in case (ii).

To prove (2.2) subtract the equations (1.1) for ϕ_1 and ϕ_2 and apply the mean value theorem obtaining

$$-\mu(\phi_1'(t,\mu) - \phi_2'(t,\mu)) = (\phi_1(t,\mu) - \phi_2(t,\mu)) \int_{-\infty}^{t} a(t-s)F_1(\sigma(t),\phi_1(s,\mu))ds$$

$$+ \int_{-\infty}^{t} a(t-s)F_2(\phi_2(t,\mu),\tau(s))(\phi_1(s,\mu) - \phi_2(s,\mu))ds ,$$

where $\sigma(t)$ is between $\phi_1(t,\mu)$ and $\phi_2(t,\mu)$ and $\tau(s)$ is between $\phi_1(s,\mu)$ and $\phi_2(s,\mu)$. Let $t = t_0$ be the last point for which $\phi_1(t,\mu) \geq \phi_2(t,\mu)$; i.e. $\phi_1(t_0,\mu) = \phi_2(t_0,\mu)$, while $\phi_1(t,\mu) \geq \phi_2(t,\mu)$

for $-\infty < t < t_0$. Since $g_1(t) \geq g_2(t)$, $-\infty < t \leq 0$, it is clear that $t_0 \geq 0$. Both integrals exist since $a \in L^1(0,\infty)$, $F \in C^1$, and ϕ_1, ϕ_2 satisfy (2.1) (or (2.1')). From the definition of t_0 and $F_2 < 0$ one has $-\mu(\phi_1'(t_0,\mu) - \phi_2'(t_0,\mu)) \leq 0$, with the equality sign holding if and only if $g_1 \equiv g_2$ on $(-\infty,0]$. Since $\mu > 0$ this implies $\phi_1'(t_0,\mu) - \phi_2'(t_0) > 0$ for $g_1 \not\equiv g_2$ and this is impossible.

To prove (2.3) put $z(t) = \phi(t,\mu_1) - \phi(t,\mu_2)$. Then from (1.1), (1.2) one has $z(0) = 0$ and

$$z'(0) = \left(-\frac{1}{\mu_1} + \frac{1}{\mu_2}\right) \int_{-\infty}^{0} a(-s)F(g(0),g(s))ds > 0 .$$

By continuity suppose that there exists $T > 0$ such that $z(t) > 0$ $(0 < t < T)$ and $z(T) = 0$. Then

$$z'(T) = -\frac{1}{\mu_1} \int_{-\infty}^{T} a(T-s)F(\phi(T,\mu_1),\phi(s,\mu_1))ds$$

$$+ \frac{1}{\mu_2} \int_{-\infty}^{T} a(T-s)F(\phi(T,\mu_2),\phi(s,\mu_2))ds .$$

But by (2.1) $\int_{-\infty}^{T} a(T-s)F(\phi(T,\mu_i),\phi(s,\mu_i))ds > 0$, $(i = 1,2)$. Therefore $\mu_1 > \mu_2$ implies that

$$z'(T) > \frac{1}{\mu_1} \int_{-\infty}^{T} a(T-s)\{F(\phi(T,\mu_2),\phi(s,\mu_2)) - F(\phi(T,\mu_1),\phi(s,\mu_1))\}ds ,$$

which by the definition of T and $\phi(t,\mu_1) = \phi(t,\mu_2) = g(t)$ on $(-\infty,0]$ yields

$$(3.20)\qquad z'(T) > \frac{1}{\mu_1}\int_0^T a(T-s)\{F(\phi(T,\mu_2),\phi(s,\mu_2)) - F(\phi(T,\mu_2),\phi(s,\mu_1))\}ds\ .$$

Applying the mean value theorem in (3.20) gives

$$(3.21)\qquad z'(T) > \frac{1}{\mu_1}\int_0^T a(T-s)F_2(\phi(T,\mu_2),\zeta(s))(\phi(s,\mu_2) - \phi(s,\mu_1))ds\ ,$$

when $\zeta(s)$ is between $\phi(s,\mu_2)$ and $\phi(s,\mu_1)$. Since $a(T-s) > 0$ and $F_2 < 0$ by assumption and since $\phi(s,\mu_2) - \phi(s,\mu_1) < 0$ on $0 < s < T$ by the definition of T, (3.21) implies that $z'(T) > 0$. Therefore there cannot exist a $T > 0$ such that $z(T) = 0$; this proves (2.3), and completes the proof of Theorem 1.

4. Proof of Theorem 2 and Corollary 2.1. We observe first that if ϕ_0 is a (continuous) solution of (1.6) for $t \geq 0$, then $\phi_0(0) = \phi_0(0^+)$ must satisfy

$$0 = \int_{-\infty}^{0} a(-s)F(\phi_0(0), g(s))ds \; . \tag{4.1}$$

This integral clearly exists for any number $\phi_0(0)$ since $a \in L^1(0,\infty)$ and (H_F), (H_g) hold; moreover by (H_F), the integral in (4.1) is a continuous, strictly increasing function of (the parameter) $\phi_0(0)$. The integral is negative if one chooses $\phi_0(0) = 1$; it is positive if one chooses $\phi_0(0) = g(0)$. Therefore, there exists a unique number $\phi_0(0) = y_0$ for which the integral in (4.1) vanishes and clearly $1 < y_0 < \phi_0(0)$. This proves (2.5).

We next observe that if ϕ_0 is a continuous solution of (1.6) on $0 \leq t \leq T < \infty$, $T > 0$, for which $\phi_0(0) = y_0$, then ϕ_0 is unique. For if $u, v \in C[0,T]$, $u(0) = v(0) = y_0$, $u(t) = v(t) = g(t)$ on $-\infty < t < 0$, and u, v satisfy (1.6) on $[0,T]$, then by (H_F) and the mean value theorem we have

$$\begin{aligned} 0 = (u(t) - v(t))\Big[\int_{-\infty}^{0} & a(t-s)F_1(\xi_1(t), g(s))ds \\ & + \int_0^t a(t-s)F_1(\xi_2(t), \eta_1(s))ds\Big] \\ & \quad + \int_0^t a(t-s)F_2(\xi_3(t), \eta_2(s))(u(s) - v(s))ds, \quad 0 \leq t \leq T \; , \end{aligned} \tag{4.2}$$

for some $\xi_1(t)$, $\xi_2(t)$, $\xi_3(t)$ between $u(t)$ and $v(t)$, and for some $\eta_1(s)$, $\eta_2(s)$ between $u(s)$ and $v(s)$. But then $a \in L^1(0,\infty)$, the continuity of F_1, F_2, $F_1 > 0$, and the continuity of u, v on $[0,T]$, together with Gronwall's inequality applied to (4.2), imply that $u(t) \equiv v(t)$, $0 \le t \le T$.

We next deduce existence of solutions of (1.6) on $[0,T]$, $T > 0$ arbitrary, as follows. Let $0 < \varepsilon \le t \le T < \infty$ and let $\{\mu_n\}_{n=0}^{\infty}$ be an arbitrary sequence with $\mu_{n+1} < \mu_n$, $\lim_{n\to\infty} \mu_n = 0$, and consider the sequence $\{\phi(t,\mu_n)\}_{n=0}^{\infty}$ of solutions of the initial value problem (1.1), (1.2) on $\varepsilon \le t \le T$. By Theorem 1, $1 < \phi(t,\mu_n) < g(0)$, and $\phi(t,\mu_{n+1}) < \phi(t,\mu_n)$, $\varepsilon \le t \le T$, $n = 0,1,\ldots$; therefore, $z(t) = \lim_{n\to\infty} \phi(t,\mu_n)$, $\varepsilon \le t \le T$, exists. Moreover, from

$$|z(t+h) - z(t)| \le |z(t+h) - \phi(t+h,\mu_n)|$$

$$+ |\phi(t+h,\mu_n) - \phi(t,\mu_n)| + |\phi(t,\mu_n) - z(t)|$$

for $t, t+h \in [\varepsilon,T]$, and from the continuity of $\phi(t,\mu)$, it follows that the limit function z is continuous on $[\varepsilon,T]$. In fact, by Dini's theorem, the sequence $\{\phi(t,\mu_n)\}$ converges to $z(t)$ uniformly on $[\varepsilon,T]$. We also observe that $\lim_{\varepsilon\to 0^+} z(\varepsilon)$ exists. The next task is to show that $z(t)$, with $z(0) = \lim_{\varepsilon\to 0^+} z(\varepsilon)$, satisfies equation (1.6) on $0 \le t \le T$. It is easier to do this by using the estimate (2.16) of Theorem 5 for $-\mu\phi'(t,\mu)$, $0 \le t \le T$; we shall only indicate below how to carry out the passage to the limit in (1.1) without the estimate (2.16). Consider the

equation (1.1) with μ replaced by μ_n, y replaced by $\phi(t,\mu_n)$. We have

$$(4.3)\quad \begin{cases} -\mu_n\phi'(t,\mu_n) = \int_{-\infty}^{t} a(t-s)F(\phi(t,\mu_n),\phi(s,\mu_n))ds, \quad 0 \le t \le T, \\ \phi(t,\mu_n) = g(t) \quad (-\infty < t \le 0)\ ; \end{cases}$$

from (2.16), $\lim_{n\to\infty} \mu_n\phi'(t,\mu_n) = 0$, $0 \le t \le T$. On the right hand side of (4.3) we use $a \in L^1(0,\infty)$, the continuity of F, and the uniform convergence of $\{\phi(t,\mu_n)\}$ to conclude that

$$\lim_{n\to\infty} \int_{-\infty}^{t} a(t-s)F(\phi(t,\mu_n),\phi(s,\mu_n))ds = \int_{-\infty}^{t} a(t-s)F(z(t),z(s))ds \quad (0 \le t \le T).$$

This shows that $z(t)$ satisfies the reduced equation (1.6) on $0 \le t \le T$. By the uniqueness of continuous solutions of (1.6) we identify $z(t)$ with $\phi_0(t)$. One can avoid using the estimate (2.16) by integrating (4.3) from 0 to t, using the boundedness of the functions $\phi(t,\mu_n)$ (by Theorem 1), and by passing to the limit in the integrated form of the equation, justified by the assumptions; we omit the details. Since $T > 0$ is arbitrary, this completes the existence and uniqueness of a continuous solution ϕ_0 of (1.6) on $0 \le t < \infty$.

To establish the remaining conclusions of Theorem 2, we first assume that $\phi_0 \in C^1[0,\infty)$. Then differentiation of (1.6), justified by (H_a), (H_F), (H_g), yields

$$(4.4)\quad 0 = \int_{-\infty}^{t} a'(t-s)F(\phi_0(t),\phi_0(s))ds + \phi_0'(t)\int_{-\infty}^{t} a(t-s)F_1(\phi_0(t),\phi_0(s))ds,$$

for $0 \leq t < \infty$, where it is useful to note that the coefficient of $\phi_0'(t)$ in (4.4) is strictly positive. It is evident that if ϕ_0 is a solution of (4.4) on $0 \leq t < T$, satisfying the initial condition

$$(4.5) \qquad \phi_0(t) = \begin{cases} g(t) & (-\infty < t < 0) \\ y_0 & (t = 0), \end{cases}$$

where y_0 is defined below (4.1), then ϕ_0 will be a continuous solution of the reduced equation (1.6) on $[0, T)$, and by uniqueness of continuous solution of (1.6) it will be the only such solution of (1.6). To establish the complete equivalence of the reduced equation (1.6) and the initial value problem (4.4), (4.5), one has to show that any continuous solution ϕ_0 of (1.6) is differentiable. This can be accomplished directly by forming difference quotients in (1.6), applying the mean value theorem and showing that one can pass to the limit thus arriving again at (4.4). Since for any continuous solution of (1.6) each integral in (4.4) defines a continuous function under our assumptions, and since the coefficient of $\phi_0'(t)$ is bounded away from zero, it follows that $\phi_0'(t)$ is continuous. This completes the discussion of the equivalence of (1.6) and (4.4), (4.5) which enables us to deduce the remaining conclusions of Theorem 2 concerning solutions of (1.6) by studying (4.4), (4.5).

A simple calculation shows that in the special case $a(t) \equiv Ae^{-\lambda t}$, $A > 0$, $\lambda > 0$, we have $\phi_0'(t) \equiv 0$ for $t > 0$ (from (4.4)), so that $\phi_0(t) \equiv y_0$ for $t \geq 0$, and this proves the remark below (2.6).

If $a(t) \not\equiv Ae^{-\lambda t}$, $A > 0$, $\lambda > 0$, we wish to show that (2.6) is satisfied. From (4.4) we first have

$$(4.6) \qquad -\phi_0'(0) = \frac{\int_{-\infty}^{0} a'(-s)F(y_0, g(s))ds}{\int_{-\infty}^{0} a(-s)F_1(y_0, g(s))ds} .$$

By (H_a), (H_F), (H_g) the denominator in (4.6) is positive. To determine the sign of the numerator we note first that by the definition of y_0

$$(4.7) \qquad \int_{-\infty}^{0} a(-s)F(y_0, g(s))ds = 0 ,$$

and by the log convexity of a, repeating the argument of Theorem 1 showing that the integral $I(\alpha) > 0$ in (3.10) (here we take $\alpha = 0$, $\phi(\alpha,\mu) = y_0$, $\phi(s,\mu) = g(s)$, and define $-\beta$, $\beta > 0$, by the relation $g(-\beta) = y_0$), we find

$$(4.8) \qquad \int_{-\infty}^{0} a'(-s)F(y_0, g(s))ds > 0 .$$

Therefore, by (4.6), (4.7), (4.8), one has $\phi_0'(0) < 0$.

Since $\phi_0 \in C^1(0,\infty)$ by continuity one has $\phi_0'(t) < 0$, $0 \leq t < \alpha$, for some $\alpha > 0$. We claim, as in the analogous part of the proof of Theorem 1 (see the proof of (3.2)), that

$$(4.9) \qquad \phi_0(t) > 1 \qquad (0 \leq t \leq \alpha) .$$

Since $\phi_0(0) = y_0 > 1$ one has by continuity that (4.9) holds on some

interval to the right of $t = 0$. Letting $0 < t_1 \le \alpha$ be the first point at which $\phi_0(t_1) = 1$, and using $y_0 > \phi_0(t) > 1$ or $0 < t < t_1$, one computes $\phi_0'(t_1)$ from (4.4), and one finds easily that $\phi_0'(t_1) > 0$, which is impossible. This proves (4.9).

We next show that

$$\phi_0'(t) < 0 \qquad (0 \le t < \infty) . \tag{4.10}$$

Here the procedure differs somewhat from the analogous part of the proof of Theorem 1 in that we do not need to compute ϕ_0''. Suppose $t = \alpha$ is the first point at which

$$\phi_0'(\alpha) = 0 \quad \text{and} \quad \phi_0'(t) < 0 \qquad (0 \le t < \alpha) . \tag{4.11}$$

Moreover, by (4.4) we have

$$\phi_0'(\alpha) = - \frac{\int_{-\infty}^{0} a'(\alpha - s)F(\phi_0(\alpha), g(s))ds + \int_0^{\alpha} a'(\alpha - s)F(\phi_0(\alpha), \phi_0(s))ds}{\int_{-\infty}^{0} a(\alpha - s)F_1(\phi_0(\alpha), g(s))ds + \int_0^{\alpha} a(\alpha - s)F_1(\phi_0(\alpha), \phi_0(s))ds} . \tag{4.12}$$

The denominator in (4.12) is clearly positive by (H_a), (H_F), (H_g). The balancing argument, involving (1.6) in the $t = \alpha$ and the log convexity of a which has been previously employed in the proof of (3.10) and (4.8), shows that

$$\int_{-\infty}^{0} a'(\alpha - s)F(\phi_0(\alpha), g(s))ds + \int_0^{\alpha} a'(\alpha - s)F(\phi_0(\alpha), \phi_0(s))ds > 0 .$$

Therefore $\phi_0'(\alpha) < 0$, contradicting (4.11) and proving (4.10). This completes the proof of (2.6).

The proof of property (2.7) is similar to that of (2.2) in Theorem 1 and is omitted.

To prove property (2.8) we observe first that since $\phi(0,\mu) = g(0)$ and since $\phi_0(0) = y_0$, (2.8) is true at $t = 0$ by (2.5), and therefore, by continuity, (2.8) is true in some interval for $t \geq 0$. Suppose that $\phi(t,\mu) - \phi_0(t)$ is zero for the first time at $t = t_0 > 0$ and $\phi(t,\mu) - \phi_0(t) > 0$ $(0 \leq t < t_0)$. Then from (1.1), (1.6) at $t = t_0$, one obtains by subtraction

$$(4.13) \quad -\mu\phi'(t_0,\mu) = \int_0^{t_0} a(t_0 - s)[F(\phi(t_0,\mu),\phi(s,\mu)) - F(\phi_0(t_0),\phi_0(s))]\,ds .$$

Applying the mean value theorem to the difference under the integral (note that $\phi(t_0,\mu) = \phi_0(t_0)$ by the definition of t_0) and using $F_2(y,z) < 0$ and $\phi(s,\mu) - \phi_0(s) > 0$ $(0 \leq s < t_0)$, it is evident that the right-hand side of (4.13) is negative. This implies that $\phi'(t_0,\mu) > 0$ which contradicts (2.1) and proves (2.8); this completes the proof of Theorem 2.

To establish the conclusion (2.10) of Corollary 2.1 let $\eta > 0$ be given. Using (2.4), (2.8), (2.9) choose $T > 0$ so large that $|\alpha(\mu) - \phi(t,u)| < \frac{\eta}{3}$ and $0 < \phi_0(t) - \alpha_0 < \frac{\eta}{3}$ for $t \geq T$. By the convergence of $\phi(t,u)$ to $\phi_0(t)$ as $\mu \to 0^+$ on $0 < t < \infty$ choose $\mu > 0$ sufficiently small that $0 < \phi(T,\mu) - \phi_0(T) < \frac{\eta}{3}$. Then

$$0 < \alpha(\mu) - \alpha_0 \leq |\alpha(\mu) - \phi(T,\mu)| + (\phi(T,\mu) - \phi_0(T)) + (\phi_0(T) - \alpha_0) < \eta ,$$

for $T > 0$ sufficiently large and $\mu > 0$ sufficiently small, proving (2.10).

5. Proof of Theorem 3. In view of the monotonicity property (2.7) of solutions of (1.6) with respect to the function g, it suffices to prove the result for the function g given by

$$(5.1) \qquad g(t) = \begin{cases} 1+\delta & \text{if } -\eta \le t \le 0 \\ 1 & \text{if } t < -\eta, \ \delta > 0, \ \eta > 0 . \end{cases}$$

Note that any such discontinuous function g can be approximated by arbitrarily smooth functions g satisfying assumption (H_g). Since $1 \le g(t) \le 1+\delta$, $-\infty < t \le 0$, property (2.6) implies that $1 \le \phi_0(t) \le 1+\delta$, $0 \le t < \infty$. This means that the arguments of F in (1.6) are close to 1, if $\delta > 0$ is sufficiently small which will be the case in what follows. For this reason we assume consistent with (H_F) and without loss of generality that

$$(5.2) \qquad F_1(1,1) = 1, \quad F_2(1,1) = -1 ;$$

note that $F(x,x) = 0$ $(x > 0)$ implies that $F_1(x,x) = -F_2(x,x)$.

Substituting (5.1) into (1.6) yields (note $a \in L^1(0,\infty)$),

$$(5.3) \qquad F(\phi_0(t),1)\int_{-\infty}^{-\eta} a(t-s)ds + F(\phi_0(t),1+\delta)\int_{-\eta}^{0} a(t-s)ds$$
$$+ \int_0^t a(t-s)F(\phi_0(t),\phi_0(s))ds = 0 .$$

Using (5.2), the mean value theorem, and $F \in C^1(\mathbb{R}\times\mathbb{R})$ yields

$$
(5.4)\qquad \begin{cases} (\phi_0(t) - 1 \int_{-\infty}^{-\eta} a(t-s)ds + (\phi_0(t) - 1 - \delta) \int_{-\eta}^{0} a(t-s)ds \\[1ex] \qquad + o(\delta) \int_{-\infty}^{0} a(t-s)ds + \int_{0}^{t} a(t-s)(\phi_0(t) - \phi_0(s))ds \\[1ex] \qquad + \int_{0}^{t} a(t-s)[o(\phi_0(t) - \phi_0(s))]\, ds = 0 \,, \end{cases}
$$

where (by $a \in L^1(0,\infty)$, $a(t) > 0$) the terminology $w(t) = o(\delta) \int_{-\infty}^{0} a(t-s)ds$ means that for every $\varepsilon > 0$ one has $|w(t)| \leq \varepsilon\delta \int_{t}^{\infty} a(\xi)d\xi$ for $t \geq 0$ and for $\delta > 0$ sufficiently small. An equivalent form of (5.4) is

$$
(5.5)\qquad (\phi_0(t) - 1) \int_{0}^{\infty} a(\xi)d\xi - \int_{0}^{t} a(t-s)(\phi_0(s) - 1)ds = \delta \int_{t}^{t+\eta} a(\xi)d\xi
$$
$$
+ o(\delta) \int_{t}^{\infty} a(\xi)d\xi + \int_{0}^{t} a(t-s)[o(\phi_0(t) - \phi_0(s))]\, ds \,.
$$

Since $1 \leq \phi_0(t) \leq 1 + \delta$ on $0 \leq t < \infty$, (H_F) implies that the first term in (5.3) is positive, while the second and third terms are negative. Thus the third term in (5.3) must be in modulus less than the first term. This in turn implies that the last term in (5.5) (or the last integral in (5.4)) must be $o(\delta) \int_{t}^{\infty} a(\xi)d\xi$. Therefore, putting $z(t) = \phi_0(t) - 1$ in (5.5) yields the equivalent equation

$$
(5.6)\qquad z(t)A - a * z(t) = \delta \int_{t}^{t+\eta} a(s)ds + o(\delta \int_{t}^{\infty} a(s)ds) \quad (0 \leq t < \infty) \,,
$$

where $*$ denotes the convolution and $A = \int_{0}^{\infty} a(s)ds$.

We shall now apply a simple Tauberian theorem for (real) Laplace transforms [20; Theorem 4.3, p. 192] to solutions of (5.6). Put

$$\psi(t) = \int_t^{t+\eta} a(s)ds, \quad \omega(t) = \int_t^{\infty} a(s)ds , \tag{5.7}$$

$$\hat{z}(p) = \int_0^{\infty} e^{-pt} z(t)dt \qquad (p > 0) . \tag{5.8}$$

Since z is bounded and continuous on $[0,\infty)$, and since $a \in L^1(0,\infty)$, $\hat{z}(p)$ as well as $\hat{\psi}(p)$ and $\hat{\omega}(p)$ exist for $p > 0$. Noting that $A = \hat{a}(0)$ and that multiplication of (5.6) by e^{-pt} and integration preserves the o relation when p is real, we obtain on solving for $\hat{z}(p)$

$$\hat{z}(p) = \frac{\delta\hat{\psi}(p) + o(\delta)\hat{\omega}(p)}{\hat{a}(0) - \hat{a}(p)} \quad (p > 0) . \tag{5.9}$$

By assumptions $(H_a$, (2.11), and Lebesgue's dominated convergence theorem, and integration by parts we also have

$$\hat{w}(0) = \int_0^{\infty}\int_t^{\infty} a(s)dsdt = \int_0^{\infty} sa(s)ds , \tag{5.10}$$

$$\hat{\psi}(0) = \int_0^{\infty}\int_t^{t+\eta} a(s)dsdt \le \int_0^{\infty}\int_t^{\infty} a(s)dsdt = \int_0^{\infty} sa(s)ds . \tag{5.11}$$

Moreover, by Fubini's theorem and (2.11)

$$\frac{d}{dp}\hat{a}(p) = -\int_0^{\infty} te^{-pt}a(t)dt \qquad (p \ge 0) , \tag{5.12}$$

so that by the mean value theorem and (2.11)

$$\hat{a}(0) - \hat{a}(p) \sim p\int_0^{\infty} ta(t)dt \qquad (p \to 0^+) . \tag{5.13}$$

Hence using (5.10), (5.11), (5.13), letting $p \to 0^+$ in (5.9), and taking $\delta > 0$ sufficiently small, there exists a constant $K(\delta) > 0$ such that

$$\hat{z}(p) \sim \frac{K(\delta)}{p} \qquad (p \to 0^+) . \tag{5.14}$$

The above mentioned Tauberian theorem then implies

$$z(t) \sim K(\delta) > 0 \qquad (t \to +\infty) , \tag{5.15}$$

and by the definition of z (5.15) yields

$$\phi_0(t) \sim 1 + K(\delta) > 1 \qquad (t \to +\infty) ,$$

for $\delta > 0$ sufficiently small. This completes the proof of Theorem 3.

6. Proof of Theorem 4. Using (2.13), equation (1.6) may be written in the form

$$(6.1) \qquad Ay(t) - a * y(t) = \int_{-\infty}^{0} a(t-s)g(s)ds \qquad (0 \le t < \infty),$$

where $A = \int_0^\infty a(s)ds$ and $a * y(t) = \int_0^t a(t-s)y(s)ds$. Letting, as in the proof of Theorem 3, $y(t) = z(t) - 1$,

(6.1) becomes the linear Volterra equation

$$(6.2) \qquad Az(t) - a * z(t) = G(t) \qquad (0 \le t < \infty),$$

where

$$(6.3) \qquad G(t) = \int_{-\infty}^{0} a(t-s)(g(s) - 1)ds \qquad (0 \le t < \infty).$$

We proceed as in the proof of Theorem 3, but taking any $g \in (H_g)$, rather than the special g of (5.1). Taking the Laplace transform and noting that $A = \hat{a}(0)$ we obtain

$$(6.4) \qquad \hat{z}(p) = \frac{\hat{G}(p)}{\hat{a}(0) - \hat{a}(p)}.$$

By Theorem 2 $\lim_{t \to \infty} z(t) = z_\infty$ exists and $z_\infty \ge 0$. If $z_\infty > 0$, a standard Abelian theorem for Laplace transforms [20, Cor. 1b, p. 182] states that $\hat{z}(p) \sim \frac{z_\infty}{p}$ $(p \to 0^+)$. We shall apply this result to (6.4) for p real. Since by (H_g) $g(t) \ge 1$ $(-\infty < t \le 0)$, and since g is nondecreasing with $g(t) > 1$ near $t = 0$, we have by Fubini's theorem

$$(6.5)\qquad 0 < \hat{G}(p) = \int_0^\infty e^{-pt} \int_{-\infty}^0 a(t-s)(g(s)-1)\,ds\,dt$$

$$= \int_{-\infty}^0 (g(s)-1) \int_0^\infty e^{-pt} a(t-s)\,dt\,ds \qquad (p > 0)$$

Let $\varepsilon > 0$ be given; choose a number $N = N(\varepsilon) > 0$, and using (H_g) divide the interval $(-\infty, 0]$ into two parts, such that

$$(6.6)\qquad g(s) - 1 \le \varepsilon \qquad (s \le -N)\ .$$

Then (6.6) used in (6.5) yields

$$0 < \hat{G}(p) \le \varepsilon \int_{-\infty}^0 e^{-ps} \int_{-s}^\infty e^{-p\theta} a(\theta)\,d\theta\,ds + O(1) \qquad (p \to 0^+)\ ,$$

or equivalently

$$(6.7)\qquad 0 < \hat{G}(p) \le \varepsilon \int_0^\infty e^{p\sigma} \int_\sigma^\infty e^{-p\theta} a(\theta)\,d\theta\,d\sigma + O(1) \qquad (p \to 0^+)\ .$$

By an integration by parts and $a \in L^1(0,\infty)$ we also have

$$(6.8)\qquad \hat{a}(0) - \hat{a}(p) = \int_0^\infty (1 - e^{-pt}) a(t)\,dt = \int_0^\infty (1 - e^{-pt}) e^{pt} e^{-pt} a(t)\,dt$$

$$= p \int_0^\infty e^{pt} \int_t^\infty e^{-p\theta} a(\theta)\,d\theta\,dt > 0 \qquad (p > 0)\ .$$

Therefore, (6.7), (6.8) substituted into (6.4) yields

$$(6.9)\qquad \hat{z}(p) \le \frac{\varepsilon \int_0^\infty e^{pt} \int_t^\infty e^{-p\theta} a(\theta)\,d\theta\,dt + O(1)}{p \int_0^\infty e^{pt} \int_t^\infty e^{-p\theta} a(\theta)\,d\theta\,dt} \qquad (p \to 0^+)\ .$$

But by (2.14)

$$\lim_{p \to 0^+} \int_0^\infty e^{pt} \int_t^\infty e^{-p\theta} a(\theta) d\theta dt = +\infty ,$$

and this fact used in (6.9) shows that for any $\varepsilon > 0$

$$\hat{z}(p) \leq \frac{\varepsilon}{p} + o(\frac{1}{p}) \qquad (p \to 0^+) ,$$

since $\varepsilon > 0$ is arbitrary, $z_\infty > 0$ is impossible. This completes the proof of Theorem 4.

7. Proof of Theorem 5 and Corollary 5.1. Let ϕ be the solution of (1.1), (1.2). We return to equation (3.8) obtained by differentiating (1.1) and we write (3.8) in the form

$$-\mu\phi''(t,\mu) = G(t,\mu)\phi'(t,\mu) + f(t,\mu) \quad (0 \leq t < \infty) , \tag{7.1}$$

where

$$G(t,\mu) = \int_{-\infty}^{t} a(t-s)F_1(\phi(t,\mu),\phi(s,\mu))ds , \tag{7.2}$$

$$f(t,\mu) = \int_{-\infty}^{t} a'(t-s)F(\phi(t,\mu),\phi(s,\mu))ds . \tag{7.3}$$

Since $F_1 > 0$, $a \in L^1(0,\infty)$, $a(t) > 0$ $(0 \leq t < \infty)$, and since ϕ satisfies (2.2), (2.4), (2.6), we have by (H_g),

$$0 < \gamma A \leq G(t,\mu) \leq \Gamma A \quad (0 \leq t < \infty,\ \mu > 0) , \tag{7.4}$$

where

$$\begin{cases} \gamma = \inf\limits_{S} F_1(y,z),\ \Gamma = \sup\limits_{S} F_1(y,z),\ S = [1,g(0)] \times [1,g(0)] , \\ \\ A = \int_0^{\infty} a(s)ds . \end{cases} \tag{7.5}$$

We next show that there exists a constant $K > 0$, independent of μ, such that

$$|f(t,\mu)| \leq K \int_t^{\infty} a(s)ds \quad (0 \leq t < \infty,\ \mu > 0) . \tag{7.6}$$

Write $f(t,\mu) = I_1 + I_2$, where

$$I_1 = \int_{-\infty}^{0} a'(t - s)F(\phi(t,\mu), g(s))ds$$

$$I_2 = \int_{0}^{t} a'(t - s)F(\phi(t,\mu), \phi(s,\mu))ds .$$

Consider I_2 first and recall, from the proof of Theorem 1, that $F(\phi(t,\mu), \phi(s,\mu)) < 0$ $(0 \leq s < t)$. Since $a' < 0$, one has on letting $\sigma(t - s) = \frac{a'(t - s)}{a(t - s)}$ and on using the log convexity of a, that

$$(7.7) \qquad 0 < I_2 \leq (-\sigma(0)) \left| \int_{0}^{t} a(t - s)F(\phi(t,\mu), \phi(s,\mu))ds \right| .$$

But now (1.1), conclusion (2.1) of Theorem 1, together with a simple consideration of signs of the terms on the right-hand side of (1.1) shows that

$$(7.8) \qquad \left|\int_{0}^{t} a(t - s)F(\phi(t,\mu), \phi(s,\mu))ds\right| \leq \int_{-\infty}^{0} a(t - s)F(\phi(t,\mu), g(s))ds .$$

Moreover, the boundedness of ϕ, g and the continuity of F imply that

$$(7.9) \qquad \int_{-\infty}^{0} a(t - s)F(\phi(t,\mu), g(s))ds \leq \sup_{\substack{0 \leq t < \infty \\ -\infty < s \leq 0}} |F(\phi(t,\mu), g(s))| \int_{t}^{\infty} a(s)ds .$$

Combining (7.7). (7.8). (7.9) shows the existence of a constant K, independent of μ, such that I_2 satisfies the estimate (7.6). In a similar way one finds on using the log convexity of a that

$$|I_1| \leq (-\sigma(0)) \sup_{\substack{0 \leq t < \infty \\ -\infty < s \leq 0}} |F(\phi(t,\mu), g(s))| \int_{t}^{\infty} a(s)ds .$$

Combining the estimates for I_1 and I_2 establishes (7.6).

Returning to (7.1) we have by integration

$$(7.10)\qquad \frac{d}{dt}\left(-\phi'(t,\mu)\exp\left(\frac{1}{\mu}\int_0^t G(s,\mu)ds\right)\right) = \frac{f(t,\mu)}{\mu}\exp\left(\frac{1}{\mu}\int_0^t G(s,\mu)ds\right) .$$

By (7.6) one has, for $0 \le t < \infty$ and $\mu > 0$,

$$(7.11)\qquad \frac{|f(t,\mu)|}{\mu}\exp\left(\frac{1}{\mu}\int_0^t G(s,\mu)ds\right) \le \frac{K}{\mu}\exp\left(\frac{1}{\mu}\int_0^t G(s,\mu)ds + \log\int_t^\infty a(s)ds\right) .$$

Before integrating (7.10) observe that by $a, a' \in L^1(0,\infty)$ and by the log convexity of a one has

$$\frac{d}{dt}\left(\frac{1}{\mu}\int_0^t G(s,\mu)ds + \log\int_t^\infty a(s)ds\right) = \frac{G(t,\mu)}{\mu} - \frac{a(t)}{\int_t^\infty a(s)ds}$$

$$= \frac{G(t,\mu)}{\mu} + \frac{\int_t^\infty a'(s)ds}{\int_t^\infty a(s)ds} = \frac{G(t,\mu)}{\mu} + O(1)\quad (0 \le t < \infty,\ \mu > 0) .$$

A simple but tedious calculation then shows that using (7.4), (7.6) in (7.11) there exist constants $\mu_0 > 0$, $\tilde{K} = \tilde{K}(\mu_0) > 0$ such that

$$(7.12)\qquad \left|\int_0^t \frac{f(\xi,\mu)}{\mu}\exp\left(\frac{1}{\mu}\int_0^\xi G(s,\mu)ds\right)d\xi\right| \le$$

$$\tilde{K}\exp\left(\frac{1}{\mu}\int_0^t G(s,\mu)ds + \log\left(\int_t^\infty a(s)ds\right)\right)\ (0 \le t < \infty;\ 0 < \mu \le \mu_0) .$$

Then integrating (7.10) and using (7.12), and

$$-\phi'(0,\mu) = \frac{1}{\mu}\int_{-\infty}^0 a(-s)F(g(0),g(s))ds ,$$

as well as conclusion (2.1) of Theorem 1, yields the estimate (2.16), where $K_1 = \gamma A$. *This completes the proof of Theorem 5.*

Proof of Corollary 5.1. Integrating (2.16) from t to infinity and using conclusions (2.1) and (2.4) of Theorem 1 yields

$$0 < \phi(t,\mu) - \alpha(\mu) \leq \tilde{K}e^{-K_1 t/\mu} + \tilde{K}\int_t^\infty \left(\int_\xi^\infty a(s)ds\right)ds \quad (0 < \mu \leq \mu_0;\ 0 \leq t < \infty)\ ;$$

an integration by parts gives

$$0 < \phi(t,\mu) - \alpha(\mu) \leq \tilde{K}e^{-K_1 t/\mu} + \tilde{K}\int_t^\infty (s-t)a(s)ds \quad (0 < \mu \leq \mu_0;\ 0 \leq t < \infty)\ . \tag{7.13}$$

Since $a(t)$ is log convex, $\log a(t)$ is bounded below by an affine function; thus there exist constants $\alpha > 0$, $\beta > 0$ such that $a(t) \geq \alpha e^{-\beta t}$ $(0 \leq t < \infty)$. Hence given any $t_0 > 0$ there exists a constant which we again call $\mu_0 > 0$ such that the integral term in (7.13) dominates the first term for $t_0 \leq t < \infty$, $0 < \mu \leq \mu_0$; this statement remains true at $t_0 = 0$ by possibly increasing $\tilde{K}$. This, together with (7.13) establishes (2.17).

The first inequality in (2.18) follows from (2.8). To prove the second inequality in (2.18) note that

$$\begin{aligned}\phi(t,\mu) - \phi_0(t) &= \phi(t,\mu) - \alpha(\mu) + \alpha(\mu) - \phi_0(t)\\ &\leq \phi(t,\mu) - \alpha(\mu) + \alpha(\mu) - \alpha_0\ ,\end{aligned}$$

where the last step follows by Theorem 2; (2.18) now follows by estimating $\phi(t,\mu) - \alpha(\mu)$ by (2.17). This completes the proof of Corollary 5.1.

8. Proof of Theorem 6. Let ϕ_0 be the unique solution of (1.6) (see Theorem 2). Thus ϕ_0 satisfies

$$(8.1)\quad \int_{-\infty}^{0} a(t-s)F(\phi_0(t),g(s))ds + \int_0^t a(t-s)F(\phi_0(t),\phi_0(s))ds = 0 \quad (0 \le t < \infty) .$$

By $a \in L^1(0,\infty)$, (H_F), (H_g), and the boundedness of ϕ_0 there exists a constant $K > 0$ such that

$$(8.2)\qquad 0 < \int_{-\infty}^{0} a(t-s)F(\phi_0(t),g(s))ds \le K\int_t^{\infty} a(s)ds \quad (0 \le t < \infty) .$$

(The first inequality in (8.2) follows by simple consideration of signs in (8.1)). Applying (H_F) and the mean value theorem we have

$$F(\phi_0(t),\phi_0(s)) = F_1(\phi_0(s),\phi_0(s))(\phi_0(t) - \phi_0(s)) + o(\phi_0(t) - \phi_0(s)) ,$$

which by the boundedness of ϕ_0, $F_1 > 0$, and the continuity of F_1 implies that there exist constants M_1, $M_2 > 0$ such that

$$(8.3)\quad M_1(\phi_0(s) - \phi_0(t)) \le -F(\phi_0(t),\phi_0(s)) \le M_2(\phi_0(s) - \phi_0(t));\quad 0 \le s \le t < \infty .$$

Using (8.2), (8.3) in (8.1) shows that

$$(8.4)\qquad \int_0^t a(t-s)(\phi_0(t) - \phi_0(s))ds = \psi(t) \quad (0 \le t < \infty)$$

where

$$(8.5)\qquad \psi(t) = O(\int_t^{\infty} a(s)ds) .$$

Let

(8.6) $$z(t) = \phi_0(t) - \alpha_0 \quad (\alpha_0 = \lim_{t\to\infty} \phi_0(t)) .$$

Then, from (8.4), z *satisfies the linear* Volterra equation

(8.7) $$z(t)\int_0^t a(s)ds - \int_0^t a(t-s)z(s)ds = \psi(t) \quad (0 \le t < \infty)$$

writing $\int_0^t a(s)ds = \int_0^\infty a(s)ds - \int_t^\infty a(s)ds$ and *noting* that $z(t)$ is bounded shows that (8.7) may be written in the form

(8.8) $$\begin{cases} z(t)A - a * z(t) = w(t) \quad (0 \le t < \infty;\ A = \int_0^\infty a(s)ds > 0) \\[2ex] w(t) = \psi(t) + z(t)\int_t^\infty a(s)ds = O(\int_t^\infty a(s)ds), \quad 0 \le t < \infty . \end{cases}$$

We now solve (8.8) by Laplace transforms. By the argument of Theorem 3 and using $\int_0^\infty ta(t)dt < \infty$, we find that if

$$\hat{w}(0) = \int_0^\infty w(t)dt \neq 0$$

(note that this integral exists in view of the estimate satisfied by w and $ta(t) \in L^1(0,\infty)$), then $\lim_{t\to\infty} z(t) = z_\infty \neq 0$. But, by Theorem 2, $z_\infty = 0$. Therefore,

(8.9) $$\int_0^\infty w(t)dt = 0 .$$

Now integrate (8.8) over $[0,T]$, $T > 0$, obtaining

$$A\int_0^T z(s)ds - \int_0^T a * z(t)dt = \int_0^T w(t)dt \, .$$

But, by (8.9), the last equation can be written as

$$A\int_0^T z(s)ds - \int_0^T a * z(t)dt = -\int_T^\infty w(t)dt \, . \tag{8.10}$$

Interchanging the order of integration in the double integral on the left-hand side of (8.10) and using the estimate for w in (8.8), as well as $ta(t) \in L^1(0,\infty)$ yields

$$A\int_0^T z(s)ds - \int_0^T z(s)\int_0^T a(\sigma)d\sigma ds = O\left(\int_T^\infty (t - T)a(t)dt\right) \quad (0 \leq T < \infty) \, . \tag{8.11}$$

Using the definition of A and combining the two integrals on the left-hand side of (8.11) yields

$$\int_0^T z(s)\int_{T-s}^\infty a(\sigma)d\sigma ds = O\left(\int_T^\infty (t - T)a(t)dt\right) \, . \tag{8.12}$$

Finally, using the fact that z is decreasing on $0 \leq T < \infty$, we obtain from (8.12)

$$z(T)\int_0^T \int_{T-s}^T a(\sigma)d\sigma ds = O\left(\int_T^\infty (t - T)a(t)dt\right) \, ,$$

which on interchanging the order of integration yields

$$z(T)\int_0^T \sigma a(\sigma)d\sigma = O\left(\int_T^\infty (t - T)a(t)dt\right) \, . \tag{8.13}$$

This implies (2.19) and completes the proof of Theorem 6.

9. Proof of Theorem 7. By standard results the initial value problem (2.22) has a unique local solution $w = \xi(t)$ having

$$(9.1) \qquad -\xi'(0) = \int_{-\infty}^{0} a(-s)F(g(0), g(s))ds > 0 .$$

Thus ξ decreases initially. To continue the local solution we proceed similar to the proof of Theorem 1. First, we show that for as long as $\xi'(\tau) < 0$ one has

$$(9.2) \qquad \xi(\tau) > y_0 ,$$

where y_0 is (see Theorem 2) the unique value for which

$$(9.3) \qquad \int_{-\infty}^{0} a(-s)F(y_0, g(s))ds = 0 .$$

We observe that equation (2.22) may be regarded as an autonomous ordinary differential equation having the point y_0 as its only critical point. Recall from Theorem 2 that $(\xi(0) =)g(0) > y_0$. If ξ assumes the value y_0 at $\tau = \tau_1$, then by (9.2)

$$-\xi'(\tau_1) = \int_{-\infty}^{0} a(-s)F(y_0, g(s))ds = 0 ,$$

which is impossible if $\xi'(t_1) < 0$. This proves (9.2). On the other hand, $\xi'(0) < 0$ implies by continuity that $\xi'(\tau) < 0$ for $0 \leq \tau < \alpha$, $\alpha > 0$. If $\alpha > 0$ is the first point at which $\xi'(\alpha) = 0$, then

$$0 = -\xi'(\alpha) = \int_{-\infty}^{0} a(-s)F(\xi(\alpha), g(s))ds .$$

This by uniqueness of y_0 implies that $\xi(\alpha) = y_0$, and therefore, $w = \xi(t)$ and $w \equiv y_0$ would be two solutions of (2.22) through the point (α, y_0), contradicting uniqueness. Therefore,

$$\xi'(t) < 0 \ , \tag{9.4}$$

for as long as the solution exists. Now (9.2), (9.4) and a standard continuation argument yield the global existence and uniqueness of the solution $w = \xi(\tau)$ of (2.22) such that

$$\xi'(\tau) < 0, \ y_0 < \xi(\tau) \leq g(0) \qquad (0 \leq \tau < \infty) \ , \tag{9.5}$$

which implies that $\lim\limits_{\tau \to \infty} \xi(\tau) = \xi_\infty$ exists and $\xi_\infty \geq y_0$.

To prove (2.23) we combine (2.22) and (9.3) obtaining

$$- \frac{d\xi}{d\tau} = \int_{-\infty}^{0} a(-s)(F(\xi(\tau), g(s)) - F(y_0, g(s))ds \quad (0 \leq \tau < \infty) \ . \tag{9.6}$$

Applying the mean value theorem and (H_F) yields the existence of a $\theta(\tau)$, $0 < \theta(\tau) < 1$, such that

$$- \frac{d\xi}{d\tau} = \int_{-\infty}^{0} a(-s)F_1(y_0 + \theta(\tau)(\xi(\tau) - y_0), g(s))ds[\xi(\tau) - y_0] \ . \tag{9.7}$$

Let $S = [y_0, g(0)] \times [1, g(0)]$, $A = \int_0^\infty a(s)ds > 0$. By (H_F)

$$\gamma = \inf_{(y,z) \in S} F_{1'}(y,z) > 0 \ .$$

Therefore using this and (9.5) in (9.7) gives the differential inequality

$$- \frac{d\xi}{d\tau} \geq \gamma A(\xi(\tau) - y_0) \qquad (0 < \tau < \infty) \ . \tag{9.8}$$

Integrating (9.8) and using the initial condition $\xi(0) = g(0)$ yields the second statement in (2.23), which, together with the existence of $\lim_{\tau\to\infty} \xi(\tau) = \xi_\infty$, also implies the first statement in (2.23).

Applying the transformation (2.21) it is clear that $y(t) = \xi(t/\mu)$ is the unique solution of the initial value problem (2.20) for $\mu > 0$, $0 \le t < \infty$. Thus

$$
(9.9) \qquad \begin{cases} -\mu \dfrac{d}{dt}\xi(\dfrac{t}{\mu}) = \displaystyle\int_{-\infty}^{0} a(-s)F(\xi(\dfrac{t}{\mu}), g(s))ds \quad (\mu > 0,\ 0 < t < \infty) \\ \xi(0) = g(0) . \end{cases}
$$

Let $\phi(t,\mu)$ be the unique solution of (1.1), (1.2) on $0 \le t < \infty$, $\mu > 0$ (see Theorem 1), which can be written in the form

$$
\begin{aligned}
(9.10) \qquad -\mu \frac{d}{dt}\phi(t,\mu) &= \int_{-\infty}^{0} a(-s)F(\phi(t,\mu), g(s))\,ds \\
&+ t\int_{-\infty}^{0} a'(-s+\theta(t,s)t)F(\phi(t,\mu), g(s))ds \\
&+ \int_{0}^{t} a(t-s)F(\phi(t,\mu), \phi(s,\mu))ds \quad (0 \le t < \infty,\ \mu > 0) ,
\end{aligned}
$$

where $0 < \theta(t,s) < 1$ comes from the application of the mean value theorem to $a(t-s) - a(-s)$. Note that since $F(\phi(t,\mu), g(s))$ is bounded and $a' \in L^1(0,\infty)$, the second integral in (9.10) clearly exists. Subtracting (9.9) from (9.10), using the mean value theorem once more, and observing that the last two terms in (9.10) are $O(t)$ as $t \to 0^+$, uniformly in $\mu > 0$, yields

$$(9.11) \qquad -\mu \frac{d}{dt}[\phi(t,\mu) - \xi(\frac{t}{\mu})] = \mathfrak{A}(t,\mu)[\phi(t,\mu) - \xi(\frac{t}{\mu})] + O(t)$$

$$(t \to 0^+, \quad \text{uniformly in} \quad \mu > 0),$$

where

$$(9.12) \qquad \mathfrak{A}(t,\mu) = \int_{-\infty}^{0} a(-S)F_1(\xi(\frac{t}{\mu}) + \theta(t,\mu)(\phi(t,\mu) - \xi(\frac{t}{\mu})), g(s))ds,$$

$0 < \theta(t,\mu) < 1$. Letting $S = [1, g(0)] \times [1, g(0)]$, $A = \int_0^\infty a(s)ds$, and using (H_a), (H_F), (H_g) implies

$$(9.13) \qquad \inf_{\substack{0 \le t \le t_0 \\ u > 0}} \mathfrak{A}(t,\mu) \ge \gamma A > 0,$$

where $\gamma = \inf_S F_1(y,z)$ is independent of μ. Using (9.13) in (9.11), integrating (9.11), and applying the initial condition $\phi(0,\mu) = \xi(0) = g(0)$, yields the existence of a constant $K = K(t_0)$, independent of μ such that

$$|\phi(t,\mu) - \xi(\frac{t}{\mu})| \le K \int_0^t \frac{s}{\mu} e^{-\frac{\gamma A}{\mu}(t-s)} ds \qquad (0 \le t \le t_0; \mu > 0).$$

Thus

$$(9.14) \qquad |\phi(t,\mu) - \xi(\frac{t}{\mu})| \le \frac{K}{\gamma A}\left[t + \frac{\mu}{\gamma A} e^{-\frac{\gamma A}{\mu} t}\right] \quad (0 \le t \le t_0, \mu > 0),$$

which is (2.24) with $\overline{K} = \frac{K}{\gamma A}$. This completes the proof of Theorem 7.

10. Proof of Theorem 8. We show that

$$w(t,\mu) = \phi_0(t) + (g(0) - \phi_0(0))\exp(-\frac{Kt}{\mu}) + \gamma(t)\mu|\log \mu| , \tag{10.1}$$

where ϕ_0 is the solution of (1.6), is an upper solution of (1.1) for $0 < t < \infty$ for suitable choices of the constant $K > 0$ and the function γ. By Appendix (B) it suffices to show that w defined by (10.1) for $0 < t < \infty$ and $w(t) = g(t)$ $(-\infty < t \leq 0)$ satisfies the integrodifferential inequality

$$-\mu w'(t,\mu) < \int_{-\infty}^{t} a(t - s)F(w(t,\mu), w(s,\mu))ds \quad (0 < t < \infty) . \tag{10.2}$$

We begin by defining γ and K in (10.2). For reasons which will become apparent below let

$$\gamma(t) = \gamma_0 \exp(K_1 \int_0^t (\int_\sigma^\infty a(\tau)d\tau)d\sigma) \quad (0 \leq t < \infty) \tag{10.3}$$

where $\gamma_0 > 0$, $K_1 > 0$ are constants specified below. The function w defined by (10.1) satisfies the inequality

$$1 \leq w(t,\mu) \leq \phi_0(0) + g(0) + \gamma_0 \exp(K_1 \int_0^\infty \int_\sigma^\infty a(\tau)d\tau d\sigma)\mu|\log \mu| \quad (0 \leq t < \infty) . \tag{10.4}$$

Let J denote the closed interval $[1, 2(\phi_0(0) + g(0))]$ and let R denote the rectangle $J \times J$. Let $M > 0$ be a constant such that

$$\max_{(y,z)\in R} \{|F(y,z)|, |F_1(y,z)|, |F_2(y,z)|, |F_{11}(y,z)|, |F_{12}(y,z)|, |F_{22}(y,z)|\} \leq M \tag{10.5}$$

and let $m > 0$ be a constant such that (see (H_F))

$$\min_{(y,z)\in R} F_1(y,z) \geq m . \tag{10.6}$$

Then for any choice of γ_0, K_1, independent of μ, one has for μ sufficiently small that the points:

$$(w(t,\mu), w(s,\mu)),\ (w(t,\mu), \phi_0(s)),\ (\phi_0(t), w(s,\mu)),\ (\phi_0(t), \phi_0(s)) \in R$$

for $0 \le s \le t < \infty$; this statement is also true for $t < 0$ since $w(t,\mu) = \phi_0(t) = g(t)$ for $t < 0$. Therefore, the values of F and its first and second partial derivatives at these points are by (10.5) bounded by M.

We define K in (10.1) by $K = \frac{1}{2} mA$, where $A = \int_0^\infty a(s)ds$.

With these definitions of K and γ it remains to verify (10.2). We begin by doing this on the interval $0 < t \le \frac{2}{K}\mu|\log\mu|$. To simplify the exposition let RHS denote the integral on the right hand side of (10.2) for t on any interval under consideration. We shall also surpress the parameter μ in $w(t,\mu)$ when no confusion arises. By the mean value theorem we have for $0 < t \le \frac{2}{K}\mu|\log\mu|$:

$$(10.7)\qquad \text{RHS} = \int_{-\infty}^{t} a(t-s)F((w-\phi_0)(t)+\phi_0(t), (w-\phi_0)(s)+\phi_0(s))ds$$

$$= (w-\phi_0)(t)\int_{-\infty}^{t} a(t-s)F_1(\xi(t),\xi(s))ds + O(\mu|\log\mu|)$$

where the O term does not exceed $\frac{2Ma(0)}{K}\mu|\log\mu|$, and $\xi(t)$ lies between $\phi_0(t)$ and $w(t)$. Thus (10.1), (10.6) used in (10.7) yields

$$(10.8)\qquad \text{RHS} \ge mA[(g(0)-\phi_0(0))e^{-Kt/\mu} + \gamma(t)\mu|\log\mu|] + O(\mu|\log\mu|)\ .$$

Choosing γ_0 in (10.3) so that $mA\gamma_0 \geq 4Ma(0)/K$ and leaving K_1 arbitrary at the moment, (10.8) becomes

(10.9) $$\text{RHS} \geq mA[(g(0) - \phi_0(0))e^{-Kt/\mu} + \frac{1}{2}\gamma_0\mu|\log\mu|] .$$

On the other hand, it follows from (10.1) and (10.3) using $\gamma'(t) > 0$ $(0 < t < \infty)$ that

(10.10) $$-\mu w'(t) \leq -\mu\phi_0'(t) + K(g(0) - \phi_0(0))e^{-Kt/\mu} \quad (t > 0) .$$

Since $-\phi_0'(t) > 0$ it follows from the choice of K by comparing (10.9), (10.10) that for μ sufficiently small the desired inequality (10.2) holds on the interval $0 < t \leq \frac{2}{K}\mu|\log\mu|$.

We next verify (10.2) on the interval $\frac{2}{K}\mu|\log\mu| \leq t \leq K_2$ where K_2 is a positive constant, independent of μ, to be determined. Observe that for $t \geq \frac{2}{K}\mu|\log\mu|$,

(10.11) $$(w - \phi_0)(t) \leq \gamma(t)\mu|\log\mu| + O(\mu^2) .$$

Thus by the mean value theorem and the fact that $w(t) = \phi_0(t) = g(t)$ $(-\infty < t < 0)$ one has

(10.12) $$\begin{aligned}\text{RHS} &= (w - \phi_0)(t)\int_{-\infty}^{t} a(t - s)F_1(\xi(t),\xi(s))ds \\ &+ \int_0^{2\mu|\log\mu|/K} a(t - s)(w - \phi_0)(s)F_2(\xi(t),\xi(s))ds \\ &+ \int_{2\mu|\log\mu|/K}^{t} a(t - s)(w - \phi_0)(s)F_2(\xi(t),\xi(s))ds \quad (t \geq \tfrac{2}{K}\mu|\log\mu|) ,\end{aligned}$$

where $\xi(t)$ is between $\phi_0(t)$ and $w(t)$. Let I_1, I_2, I_3 denote the first, second, and third integral in (10.12). Then by (10.1), (10.5), and γ nondecreasing one has

$$|I_2| \le \frac{2Ma(0)}{K}\gamma(t)\mu^2|\log\mu|^2 + \frac{Ma(0)}{K}\mu(g(0) - \phi_0(0)) , \tag{10.13}$$

and similarly

$$|I_3| \le M[\gamma(t)\mu|\log\mu| + O(\mu^2)]\int_{\frac{2\mu}{K}|\log\mu|}^{t} a(t-s)ds . \tag{10.14}$$

Note that I_2, I_3 are each negative while I_1 is positive and for $\mu > 0$ sufficiently small

$$I_1 \ge \frac{3}{4}mA\{\gamma(t)\mu|\log\mu| + O(\mu^2)\} . \tag{10.15}$$

So if K_2 is chosen *(independent of* μ) and sufficiently small, it follows from (10.12) - (10.15). that for μ sufficiently small

$$\text{RHS} \ge \frac{1}{2}mA\gamma(t)\mu|\log\mu|, \left(\frac{2\mu|\log\mu|}{K} \le t \le K_2\right) . \tag{10.16}$$

From (10.1), (10.3) one has again (10.10) valid on the interval being considered. Comparing (10.10) and (10.16) one finds that (10.2) is satisfied on the interval $\frac{2}{K}\mu|\log\mu| \le t \le K_2$, with K_2 chosen as above, if $\mu > 0$ is sufficiently small. Note that in this and the previous part of the proof we have not used the special form of $\gamma(t)$ in any significant way.

Finally, we establish (10.2) for any $t > K_2$. For any such t (10.12) is valid and the estimate for I_2 given by (10.13) holds. Rewrite I_1 in

the form

$$(10.17) \qquad I_1 = (w - \phi_0(t))\left[\int_{-\infty}^{\frac{2\mu}{K}|\log \mu|} a(t - s)F_1(\xi(t), \xi(s))ds\right]$$

$$+ (w - \phi_0(t))\left[\int_{\frac{2\mu}{K}|\log \mu|}^{t} a(t - s)F_1(\xi(t), \xi(s))ds\right] = I_{11} + I_{12} .$$

Thus (10.12) may be written as

$$\text{RHS} = I_{11} + I_{12} + I_2 + I_3 .$$

We shall combine $I_{11} + I_2$ and show that $I_{12} \geq |I_3|$. From (10.17), (10.1), (10.13), and γ nondecreasing, we find easily that for $t \geq K_2$

$$I_{11} + I_2 \geq m\gamma(K_2)\mu|\log \mu| \int_{K_2}^{\infty} a(\sigma)d\sigma$$

$$- \frac{Ma(0)}{K}(2\gamma(\infty)\mu^2|\log \mu|^2 + \mu(g(0) - \phi_0(0))) .$$

Thus if μ is sufficiently small and K_2 is not too large, there exists a constant $\delta > 0$ such that for $t \geq K_2$

$$(10.18) \qquad I_{11} + I_2 \geq \delta\mu|\log \mu| .$$

To show that

$$(10.19) \qquad I_{12} \geq |I_3| \qquad (t \geq K_2) ,$$

it suffices to show that for $t \geq K_2$, $\frac{2\mu}{K}|\log \mu| \leq s \leq t$, one has

$$(10.20) \qquad (w - \phi_0)(t)F_1(\xi(t), \xi(s)) \geq (w - \phi_0)(s)|F_2(\xi(t), \xi(s))| .$$

We write $\xi(t) = \phi_0(t) + \theta_1(w - \phi_0)(t)$, $\hat{\xi}(s) = \phi_0(s) + \theta_2(w - \phi_0)(s)$, where $0 < \theta_1, \theta_2 < 1$ in the integrals I_{12} and I_3 in (10.19). The assumption $F \in C^2(\mathbb{R}^+ \times \mathbb{R}^+)$ and the bounds (10.5) insure that for the range of the arguments of interest here the first partial derivatives of F_2/F_1 do not exceed $\frac{2M^2}{m^2}$. Using these facts, as well as $F_1(x,x) = -F_2(x,x)$ for every x, and applying the mean value theorem to F_2/F_1, it follows that we will have stablished (10.20) (and also (10.19)) if we can prove that

$$\text{(10.21)} \quad \frac{(w - \phi_0)(t)}{(w - \phi_0)(s)} \geq 1 + \frac{2M^2}{m^2} \left[|\phi_0(t) - \phi_0(s)| + |(w - \phi_0)(t) - (w - \phi_0)(s)| \right] ,$$

for $t \geq K_2$ and $\frac{2\mu}{K} |\log \mu| \leq s \leq t$. But for s,t so restricted it follows from (10.1) that

$$\text{(10.22)} \qquad \frac{(w - \phi_0)(t) - (w - \phi_0)(s)}{(w - \phi_0)(s)} = \frac{\gamma(t) - \gamma(s)}{\gamma(s)} + O(\mu^2) ,$$

and using (10.3) that

$$\text{(10.23)} \qquad \frac{\gamma(t) - \gamma(s)}{\gamma(s)} = \exp(K_1 \int_s^t \int_\sigma^\infty a(\tau) d\tau d\sigma) - 1 \geq K_1 \int_s^t \int_\sigma^\infty a(\tau) d\tau d\sigma .$$

Thus using the estimate $|\phi_0(t) - \phi_0(s)| \leq \frac{M}{m} \int_s^t a(\sigma) d\sigma$ (obtained from (4.4), see also Remark 5.1) and (10.23), in (10.21) it follows that (10.21) holds if one chooses the constant K_1 sufficiently large. This proves (10.19). Combining (10.18) and (10.19) shows that for μ sufficiently small and the constants $\delta > 0$, $\gamma_0 > 0$, $K_1 > 0$ chosen as above,

$$RHS = I_{11} + I_{12} + I_2 + I_3 \geq \delta\mu |\log \mu| \quad (\delta > 0)$$

for $t \geq K_2$. A comparison with (10.10) then shows that (10.2) is also satisfied for $t \geq K_2$. This completes the proof of Theorem 8.

Appendix (A): Statement of physical problem and formulation of mathematical model

Molten plastics commonly exhibit large elastic recovery: for example, a filament of a certain polyethylene* ('Melt 1' at 150°C), when elongated at a rate of 1 cm/sec/cm from an initial length of, say, 1 cm to a length of 55 cm and then released, will reach a final length of about 5 cm [15]. Such elastic and other rheological properties are of interest in the processing of plastics and rubber and also as examples of materials with 'memory'.

The correct equations for describing the isothermal behaviour of a given molten plastic are not yet known. One set which has been used with some success [8] for 'Melt 1' can be expressed as follows in body tensors:

$$\text{(A1)} \qquad \underline{\pi}(P,t) + p\underline{\gamma}^{-1}(P,t) = -\eta \frac{\partial \underline{\gamma}^{-1}(P,t)}{\partial t} + \int_{-\infty}^{t} a(t-s)\underline{\gamma}^{-1}(P,s)ds ;$$

$$\text{(A2)} \qquad \frac{\partial}{\partial s} \det \underline{\gamma}(P,s) = 0 \qquad (-\infty < s \le t) ;$$

$$\text{(A3)} \qquad \underline{\nabla}_t \cdot \underline{\pi}(P,t) = \rho(\underline{\alpha} - \underline{\Xi}) ;$$

$$\text{(A4)} \qquad \underline{R}\{\underline{\gamma}(P,s)\} = 0 \qquad (-\infty < s \le t) .$$

(A1) is the so-called 'rubberlike liquid' constitutive equation relating the symmetric contravariant stress tensor $\underline{\pi}(P,t)$ at particle P and time t to the values of the reciprocal $\underline{\gamma}^{-1}$ of the symmetric covariant

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*The U. S. production (8×10^9 lb) of polyethylene in 1973 exceeded that of any other one polymer.

metric tensor $\underline{\gamma}(P,s)$ at times in the interval $-\infty < s \leq t$ (see (A12) below). The material properties are determined by the nonnegative constant η and the nonnegative constants $a_1, \ldots, a_m, \tau_1, \ldots, \tau_m$, in the 'memory function'

$$\text{(A5)} \qquad a(t) = \sum_{r=1}^{m} a_r \exp(-t/\tau_r) \qquad (a_r \geq 0,\ \tau_r > 0) .$$

(See (A26) and following explanation, below).

If the integral term were omitted, (A1), with the constant volume condition (A2), would describe an incompressible Newtonian liquid of viscosity η and could be used, with the stress equation of motion (A3), to derive the Navier-Stokes equations; there would be no elastic recovery possible. $\underline{\alpha}$ and $\underline{\Xi}$ are contravariant vectors describing acceleration and body force per unit mass, respectively; ρ denotes the density. p is a scalar function of P and t, of the nature of a hydrostatic pressure, introduced in conjunction with the incompressibility condition (A2). $\underline{\nabla}_t$ in (A3) denotes the covariant derivative operator formed with $\underline{\gamma}(P,t)$, and the dot denotes contraction [8, p. 193]. (A4) expresses the vanishing of the fourth rank Riemann-Christoffel curvature tensor $\underline{R}$ constructed with $\underline{\gamma}(P,s)$; this expresses the fact that the body manifold is Euclidean at each instant s and so admits a body coordinate system that is instantaneously rectangular Cartesian at s [8, p. 202]. Equations (A1 - 4) are sufficient in number to determine, in principle, the unknown variables $\underline{\pi}$, $\underline{\gamma}$, and p when the remaining quantities (together with suitable boundary and initial conditions) are given.

When referred to an arbitrary body coordinate system $B : \{P\} \to \mathbb{R}^3$, (A1) and (A2) yield equations

$$\text{(A6)} \qquad \pi^{ij}(\xi,t) + p(\xi,t)\gamma^{ij}(\xi,t) = -\eta \frac{\partial\gamma^{ij}(\xi,t)}{\partial t} + \int_{-\infty}^{t} a(t-s)\gamma^{ij}(\xi,s)ds \ ,$$

$$\text{(A7)} \qquad \frac{\partial}{\partial s}\det \gamma_{ij}(\xi,s) = 0 \qquad (-\infty < s \leq t) \ ,$$

where ξ *in this and in similar contexts is short for* (ξ^1,ξ^2,ξ^3), the coordinates of P in B; π^{ij}, γ^{ij}, and γ_{ij} denote components of $\underline{\pi}$, $\underline{\gamma}^{-1}$, and $\underline{\gamma}$, respectively, $i,j = 1,2,3$. For an arbitrary B, (A3) yields a complicated set of equations. One can, however, always choose a B that is Cartesian at the instant t at which (A3) applies (i.e., B is such that $\gamma_{ij}(\xi,t)$ is independent of ξ for $i,j = 1,2,3$), and a space coordinate system $C : \{Q\} \to R^3$ that is rectangular Cartesian; (A3) then yields the following three partial differential equations:

$$\text{(A8)} \qquad \sum_{j,k=1}^{3}\sum \frac{\partial\pi^{kj}(\xi,t)}{\partial\xi^k}\,\frac{\partial f^i(\xi,t)}{\partial\xi^j} = \rho\left\{\frac{\partial^2 f^i(\xi,t)}{\partial t^2} - X^i\right\} \qquad (i = 1,2,3) \ .$$

The motion of the body is now described by the three equations

$$\text{(A9)} \qquad x^i = f^i(\xi,s) \qquad (i = 1,2,3) \ ,$$

where x^i denote the coordinates in C of the place Q occupied by particle P at time s. X^i denotes the components in C of the external body force per unit mass.

(A4) yields a very complicated set of nonlinear, second order partial differential equations in $\gamma_{ij}(\xi,s)$ whose solution is of the form

(A10) $$\gamma_{ij}(\xi, s) = \sum_{k=1}^{3} \frac{\partial f^k(\xi, s)}{\partial \xi^i} \frac{\partial f^k(\xi, s)}{\partial \xi^j} \qquad (i, j = 1, 2, 3)$$

for arbitrary B and rectangular Cartesian C [3, 11]. We may thus use the three functions $f^i(\xi, s)$ in place of the six functions $\gamma_{ij}(\xi, s)$ $(= \gamma_{ji}(\xi, s))$ as unknowns. On using (A10) and the matrix equation

(A11) $$[\gamma^{ij}(\xi, s)] = [\gamma_{ij}(\xi, s)]^{-1},$$

we may express (A6) and (A7) in terms of π^{ij}, f^i, and p; on substituting the resulting expressions for π^{ij} into (A8), we finally obtain three non-linear, partial-integro-differential equations, which, with the single equation resulting from (A7), yields a set of four equations for the four unknown functions f^i, p; the independent variables are ξ^i, s.

The final equations are nonlinear in f^i, p although the rubberlike liquid constitutive equation (A1) is linear in the tensors $\underline{\pi}$, $\underline{\gamma}^{-1}$. The nonlinearity comes from the constant volume condition (A2) and from the zero-curvature condition (A4) whose solution (A10) is quadratic in the unknown functions f^i. The nonlinearity arising from the products in the left-hand side of (A8) can be removed trivially by choosing B to coincide with C at time t, so that $f^i(\xi, t) = \xi^i$ and $\partial f^i / \partial \xi^j = \delta_{ij}$ at time t.

A very considerable simplification of the above equations is obtained for flow histories which are homogeneous (or uniform) under conditions in which the inertial and body force terms on the right-hand side of (A3) can be neglected. Such histories are of little or no interest in classical

hydrodynamics (where the constitutive equations are given by (A1) with $a(t) \equiv 0$) but are of fundamental importance in polymer rheology where highly viscous molten plastics can be subjected to uniform elongation in filament form or to two-way stretching in sheet form; results of carefully controlled experiments of this type can be used to test the applicability of constitutive equations such as (A1). A flow history is homogeneous if, for any two instants s, t, we have $\underline{\nabla}_t \underline{\gamma}(P, s) = 0$ [8, p. 247], from which it can be shown that $\underline{\nabla}_t \underline{\gamma}^{-1}_t(P, s) = 0$; since $\underline{\nabla}_t$ commutes with the operators $\partial/\partial t$ and $\int^t a(t - s) \ldots ds$, it follows from (A1) (taking p to be independent of P, as a trial solution) that $\underline{\nabla}_t \underline{\pi}(P, t) = 0$ (showing that the stress is homogeneous) and hence also that (A3) (with the right-hand side zero) is satisfied. It also follows from the above definition of a homogeneous flow history that a body coordinate system $B : P \rightarrow \xi$ exists that is Cartesian in every state, i.e., which is such that $\gamma_{ij}(\xi, s)$ is independent of ξ for all s [8, p. 247], and hence (A4) is satisfied. The behaviour in homogeneous flow histories with inertial and body forces neglected is thus governed by (A6) and (A7) with ξ absent, and there is no longer a need to introduce a space coordinate system or the functions f^i: one can instead use γ_{ij} or γ^{ij} as the unknowns, for example, in the case of problems involving the calculation of free elastic recovery: in such problems, the flow history (and hence $\gamma_{ij}(s)$) would be specified throughout some interval $-\infty < s < t_1$, say;

for $t_1 < s < t$, the stress would be zero, and the elastic recovery would be determined by solving the set (A6) (with $\pi^{ij} = 0$) and (A7) for $p(t)$ and $\gamma_{ij}(s)$ $(t_1 < s < t)$. These equations thus form a simultaneous system of nonlinear Volterra integral equations, in which the nonlinearity arises from the incompressibility condition.

In this paper, we consider the particular case of the above in which the specified flow history is one of simple elongation (at constant volume). The variable p can be eliminated, and the recovery behaviour is then governed by a single nonlinear Volterra equation, which we now derive.

In any B, the separation P_0P at time t between any two neighbouring particles P_0, P is given by the equation

(A12) $$(P_0P)_t^2 = \sum_i \sum_j \gamma_{ij}(\xi, t)\,\delta\xi^i\,\delta\xi^j$$

where ξ^i and $\xi^i + \delta\xi^i$ are the coordinates in B of P_0 and P. For any two times s, t, there are three material lines through any given P_0 that are mutually orthogonal at s and at t; in the strain $s \rightarrow t$, infinitesimal material line elements tangential to these three material lines at P_0 change in length by factors $\lambda_i(P_0, s, t)$ which are given by the positive roots in λ of the equation

(A13) $$\det\{\gamma_{ij}(\xi, t) - \lambda^2\gamma_{ij}(\xi, s)\} = 0 .$$

The factors λ_i are called 'principal elongation ratios'. A flow is 'shear free' if there exists a body coordinate system B that is always

orthogonal, i.e., such that $\gamma_{ij} = 0$ when $i \neq j$ [8, p. 81]. For a shear-free flow, the principal elongation ratios are given by the length changes of the coordinate lines, and the roots of (A13) are given by

$$\lambda_i^2 \gamma_{ii}(\xi, s) = \gamma_{ii}(\xi, t) \qquad (i = 1, 2, 3) . \tag{A14}$$

For a shear-free flow that is homogeneous, B is always Cartesian, i.e., the γ_{ii} are independent of ξ*. A shear-free flow is a 'simple elongation' if two principal elongation ratios, λ_2 and λ_3 say, are always equal; the ξ^1 coordinate lines are then called directions of elongation. The constant volume condition (A7) reduces to the equation $\lambda_1 \lambda_2 \lambda_3 = 1$ and hence, for simple elongation at constant volume, we have

$$\lambda_2 = \lambda_3 = \lambda_1^{-\frac{1}{2}} . \tag{A15}$$

We now consider the following problem:

$-\infty < s \leq -t_0$ Zero stress; no flow; B rectangular Cartesian; hence

$$\pi^{ij} = 0, \; \gamma_{ij} = \gamma^{ij} = \delta_{ij} . \tag{A16}$$

$-t_0 \leq s \leq 0$ Homogeneous simple elongation at constant volume and constant rate κ, i.e.,

$$\frac{d\lambda_1(-t_0, s)}{ds} = \kappa \lambda_1(-t_0, s), \; \lambda_2 = \lambda_3 = \lambda_1^{-\frac{1}{2}} . \tag{A17}$$

$0 < s \leq t$ Zero stress; free elastic recovery:

$$\pi^{ij} = 0 . \tag{A18}$$

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*And hence λ_i are independent of P_0.

We wish to calculate $\gamma_{ij}(\xi, s)$ for $0 < s < t$. As a trial solution, it is reasonable to suppose that the elastic recovery will involve a homogeneous simple elongation at constant volume with the ξ^1-coordinate lines again as directions of elongation (or contraction). For convenience, we write

$$\text{(A19)} \qquad y(s) := \lambda_1(-t_0, s) \, .$$

Since the entire flow history is a homogeneous simple elongation at constant volume, we have, from (A11), (A14), (A15), (A16), and (A19),

$$\text{(A20)} \qquad \left.\begin{aligned} \gamma^{11}(\xi, s) = y^{-2}(s), \; \gamma^{22}(\xi, s) = \gamma^{33}(\xi, s) = y(s), \\ \gamma^{ij} = 0 \quad (i \neq j) \, . \end{aligned}\right\} \; (-\infty < s \leq t)$$

From (A16) and (A17), we have

$$\text{(A21)} \qquad y(s) = \begin{cases} 1 & (-\infty < s \leq -t_0) \, , \\ \exp\{\kappa(s + t_0)\} & (-t_0 \leq s \leq 0) \, . \end{cases}$$

$y(s)$ is to be calculated for $0 < s \leq t$ so as to satisfy (A6) and (A7).

(A7) is satisfied by (A20). Using (A20) and (A18), the six equations (A6) for $t > 0$ reduce to the following two:

$$\text{(A22)} \qquad (i = j = 1) \quad p(t)y^{-2}(t) = -\eta \frac{dy^{-2}(t)}{dt} + \int_{-\infty}^{t} a(t - s)y^{-2}(s)ds \, ,$$

$$\text{(A23)} \qquad (i = j = 2, 3) \quad p(t)y(t) = -\eta \frac{dy(t)}{dt} + \int_{-\infty}^{t} a(t - s)y(s)ds \, .$$

The unknown function $p(t)$ may be eliminated by multiplying (A22) by $y^3(t)$ and then subtracting (A23). The resulting equation may be written in the form

$$\text{(A24)} \qquad -\mu \frac{dy(t)}{dt} = \int_{-\infty}^{t} a(t-s)F(y(t),y(s))ds \qquad (t > 0) ,$$

where

$$\text{(A25)} \qquad \mu := 3\eta, \; F(y,z) := (y^3/z^2) - z .$$

These are the equations used in the text above. Finally, we add some brief remarks about the physical basis and applicability of the rubberlike-liquid constitutive equation (A1) which has been discussed elsewhere [8, pp. 143, 223-236].

(A1) has been derived from two different molecular theories: the 'bead-spring' theory of Rouse and Zimm for very dilute solutions of deformable long molecules in an incompressible Newtonian solvent of viscosity η, and the network theory of Green and Tobolsky, Yamamoto, and Lodge which is developed for concentrated polymer solutions and undiluted or molten polymers. It is curious that two different molecular theories should yield constitutive equations of the same form, but the reason for this is known: the differences between the two sets of equations at the molecular level do not survive the averaging process used to go from the molecular quantities to the macroscopic quantities $\underline{\pi}$ and $\underline{\gamma}$ which appear in the constitutive equation (A1) [9]. The memory function constants a_r, τ_r are specified in terms of three unknown constants by the bead-spring theory but are not specified by the network theory, which also leaves η unspecified.

According to the network theory, the integral term in (A1) arises from the thermal motion of a network composed of long, deformable polymer molecules temporarily linked together at a few points called entanglements or temporary junctions which are assumed to be created and lost at constant rates which are unaffected by the flow history. The concentration $N(t)dt$ of network strands which were created in the interval $(0, dt)$ and are still in the network at time t (a 'strand' being that part of a polymer molecule lying between two consecutive junctions) is given by an equation of the form

$$N(t) = \sum_{r=1}^{m} C_r \exp(-t/\tau_r) \tag{A26}$$

where, for simplicity, it has been assumed that the set of all strands can be sorted into m subsets, labelled $1, 2, \ldots, m$, such that, in the r^{th} subset, all strands were created at the same rate C_r (per millilitre) and have the same probability $1/\tau_r$ per second of leaving the network [10]. The memory function in (A1) is given by the equation $a(t) = kTN(t)$, where k is Boltzmann's constant and T is the absolute temperature. Thus $a_r = kTC_r > 0$, and (A5) is proved.

According to the network theory, then, it follows that $a(t) > 0$, because there is always a nonzero concentration of strands of age t, and that $a'(t) < 0$ because strands of age $t(> 0)$ can only be lost and not created; strands are created with age 0 only. It also follows from (A5) that

$$\text{(A27)} \qquad \left.\begin{array}{l} (-1)^k a^{(k)}(t) > 0 \; ; \\ \text{and } \; a^{(k+1)}(t)/a^{(k)}(t) \; \text{ is nondecreasing} \end{array}\right\} (k = 0, 1, 2, \ldots) \, .$$

$a^{(k)}(t)$ denotes the k^{th} derivative of $a(t)$. (A27) represents the properties of $a(t)$ some of which are used in the present analysis.

The constitutive equation obtained by putting $\eta = 0$ in (A1) leads to the 'reduced' equation (1.6) and has been tested for 'Melt 1' by comparing the predictions with results of a series of experiments performed by Meissner [13,15]. The constants in (A5), with $m = 5$, were chosen to fit stress growth data in simple elongation at low rates. (A1) (with $\eta = 0$) then gave good agreement with stress growth data in simple elongation at higher rates, with elastic recovery data following elongation and following shear, and with stress growth data in shear flow [8, pp. 225-231; 2; 4], provided that the total strain from rest was limited to moderate values; at higher strains, there was serious disagreement between theory and data: the predicted stresses and the predicted recoveries were greater than the observed. The present analysis of the elongational recovery problem shows that inclusion of the term in η leads to a reduction in the predicted recovery, which is in the right direction to give better agreement with experiment. The term in η has been added to represent the possible effect of the presence of a viscous solvent (in the case of a concentrated polymer solution) or of low-molecular-weight polymer (in the case of a molten polymer). The possible effects of such a term are also of some

interest in connection with certain 'fast-strain' tests of the Gaussian network hypothesis which have recently been proposed as a possible method of testing certain of the network theory assumptions when separated from the others [8, pp. 231-236; 12]. It is recognized, however, that other modifications to (A1) are required if better agreement between all the predictions and data referred to above is to be obtained [14]. It should, perhaps, be added that the homogeneous elongation with neglect of inertial and body forces treated in the present analysis represents a reasonable idealization of the conditions of Meissner's elongation experiments on 'Melt 1': a long filament of high viscosity (5×10^5 poise) was floated on a bath of an inert oil, and the homogeneity of elongation was always checked by weighing samples into which the filament was cut after elongation; the variation of elongation ratio along the filament was about 3% or less [15].

The present analysis depends on the properties

$$F(x,x) = 0;\ \partial F(x,z)/\partial x > 0;\ \partial F(x,z)/\partial z < 0\ , \tag{A28}$$

of the function $F(y(t), y(s))$ in the integrand of (1.1). We have been able to show that there is reason to believe that these properties would be exhibited by the corresponding integrands for the same elastic recovery problem when members of a much wider class of possible constitutive equations than (A1) are considered, namely, the class of constitutive equations of the form

$$\underline{\pi}(t) \cdot \underline{\gamma}(t) + p(t)\underline{\delta} = -\eta \frac{\partial \underline{\gamma}^{-1}(t)}{\partial t} \cdot \underline{\gamma}(t) + \int_{-\infty}^{t} a(t-s) \sum_k W_k(I_1, I_2)\underline{\theta}^k(s,t)ds\ , \tag{A29}$$

where the common argument P has been omitted for brevity, W_k are arbitrary functions of the strain invariants

$$I_1 = \underline{\gamma}^{-1}(s) : \underline{\gamma}(t);\ I_2 = \underline{\gamma}^{-1}(t) : \underline{\gamma}(s)\ , \tag{A30}$$

$\underline{\delta}$ denotes the right-covariant unit tensor, and $\underline{\theta}^k = \underline{\theta} \cdot \underline{\theta} \cdots \underline{\theta}$ (k factors) where $\underline{\theta}$ denotes a right-covariant strain tensor defined by the equation

$$\underline{\theta}(s,t) = \underline{\gamma}^{-1}(s) \cdot \underline{\gamma}(t)\ . \tag{A31}$$

(A1) is obtained from (A29) on putting $W_1 = 1$, *all other* $W_k = 0$, *and* contracting from the right with $\underline{\gamma}^{-1}(t)$. (A29) has been considered by others [1, 5]. With W_1 and W_2 arbitrary, no extra generality is obtained by including terms with $k > 2$, *because higher powers of* $\underline{\theta}$ can be expressed in terms of lower powers by a tensor analogue of the Cayley-Hamilton theorem (remembering that the third invariant $\det \underline{\theta}$ is equal to 1 because of the *assumed incompressibility*).

It is a straightforward matter to show that (A29) also leads to an equation of the form (1.1) with

$$F(x,z) = xG(x/z),\ G(\alpha) = \sum_k W_k(\alpha)(\alpha^{2k} - \alpha^{-k})\ . \tag{A32}$$

Thus $F(x,x) = 0$, as required, and the two remaining properties in (A28) will hold if, and only if, $G'(\alpha) > 0\ (\alpha > 1)$. If we now put $\eta = 0$ and consider the particular "step-function" simple elongation history (at constant volume) for which the material is at rest at zero stress up to the

instant 0 when it is given an instantaneous simple elongation of magnitude $\alpha > 1$, it is easy to obtain the following expression for the tensile stress immediately after the elongation:

$$(p_{11} - p_{22})_{0+} = G(\alpha) \int_0^\infty a(u)du \quad \text{(A33)}$$

p_{ii} denote Cartesian components of stress referred to a rectangular Cartesian coordinate system with the 1-axis parallel to the direction of elongation. Since it is reasonable to regard the material in these circumstances as (momentarily) a perfectly elastic isotropic solid during the instantaneous elongation, it is reasonable to expect that the tensile stress will increase monotonically with the elongation ratio α. Since the integral in (A33) is a positive constant, it then follows that $G'(\alpha) > 0$ $(\alpha > 1)$. This completes the justification of (A28) for constitutive equations of the form (A29).

Appendix (B)

Our purpose is to collect the results on integrodifferential inequalities needed in the proof of Theorem 8. These are rather similar to classical results of this type for ordinary differential equations and Volterra equations developed e.g. in [6] and [19]. However, our situation is sufficiently different, so that it is simpler to give an independent short exposition of what is used, rather than to try to apply the known results.

In what follows let $D_-u(t)$ denote the lower left-hand Dini derivative, $D^-u(t)$ denote the upper left-hand Dini derivative, $D_+u(t)$ the lower-right hand Dini derivative, and $D^+u(t)$ the upper-right hand Dini derivative of a continuous function u. When $D^+u(t) = D_+u(t)$, we denote this common value by $u'_+(t)$, the right-hand derivative of u and similarly $u'_-(t)$ *denotes the left-hand derivative of* u.

It will be convenient to write the initial value problem (1.1), (1.2) in the form

$$\text{(B1)} \qquad \begin{cases} -\mu y'(t) = \displaystyle\int_0^t a(t-s)F(y(t),y(s))ds + f_a(y)(t) \\ y(0) = g(0) \qquad (\mu > 0;\ 0 < t < \infty) , \end{cases}$$

where

$$\text{(B2)} \qquad f_a(y)(t) = \int_{-\infty}^{0} a(t-s)F(y(t),g(s))ds .$$

Throughout this appendix we shall assume

(B3) $\quad a(t) \geq 0$, $a \in L^1(0,\infty)$, F satisfies (H_F), g satisfies (H_g).

The basic result needed is (compare Theorem 1.2.1 of [6]).

<u>Proposition 1B</u>. <u>Let the assumptions</u> (B3) <u>be satisfied</u>. <u>Let</u> $v, w \in C[[0,\gamma); \mathbb{R})$, $\gamma > 0$, <u>be given functions satisfying the following properties</u>:

(B4) $$v(0) < w(0) ,$$

(B5) $$\begin{cases} -\mu D_- v(t) \geq \int_0^t a(t-s)F(v(t), v(s)) + f_a(v)(t) & (0 < t < \gamma) , \\ -\mu D_- w(t) < \int_0^t a(t-s)F(w(t), w(s)) + f_a(w)(t) & (0 < t < \gamma) . \end{cases}$$

<u>Then</u> $v(t) < w(t) \quad (0 \leq t < \gamma)$.

<u>Proof</u>. Define the set $Z = \{t \in [0,\gamma) : w(t) \leq v(t)\}$. If Proposition 1B is false the set $Z \neq \emptyset$; let $t_1 = \text{Inf } Z$. By (B4), $t_1 = \inf Z > 0$ and

(B6) $$v(t_1) = w(t_1), \; v(t) < w(t) \quad (0 \leq t < t_1) .$$

Taking $h < 0$, $|h|$ small, one has $v(t_1 + h) < w(t_1 + h)$ and

$$\frac{v(t_1 + h) - v(t_1)}{h} > \frac{w(t_1 + h) - w(t_1)}{h} .$$

Taking the limit inferior as $h \to 0^-$ this implies

$$D_- v(t_1) \geq D_- w(t_1) ,$$

and therefore,

$$-\mu D_- v(t_1) \leq -\mu D_- w(t_1) \quad (\mu > 0) .$$

Applying this and $v(t_1) = w(t_1)$ in inequalities (B5) yields the inequality

(B7) $$\int_0^{t_1} a(t_1 - s)F(v(t_1), v(s)ds < \int_0^{t_1} a(t_1 - s)F(w(t_1), w(s))ds .$$

On the other hand, the definition of t_1 and (B6), together with the assumption $F_2 < 0$ in (H_F) implies that

$$F(v(t_1), v(s)) > F(w(t_1), w(s)) \qquad (0 \leq s < t_1) ,$$

so that, since $a(t) \geq 0$,

$$\text{(B8)} \qquad \int_0^{t_1} a(t_1 - s)F(v(t_1), v(s))ds > \int_0^{t_1} a(t_1 - s)F(w(t_1), w(s))ds .$$

Thus (B8) contradicts (B7) and the set $Z = \emptyset$. This proves Proposition 1B.

Definition. We shall say that w is an upper solution of the initial value problem (B1) on $0 < t < \gamma$ if and only if $w \in C[0, \gamma)$, $w(0) \geq g(0)$, $w'_+(t)$ exists on $(0, \gamma)$ and

$$-\mu w'_+(t) < \int_0^t a(t - s)F(w(t), w(s))ds + f_a(w)(t) \qquad (0 < t < \gamma) .$$

A similar definition holds for a lower solution with the inequality reversed. We remark that as a consequence of Theorem 2 the solution ϕ_0 of the reduced equation (1.6) (recall that it was proved in Theorem 2 that $\phi'_0(t) < 0$ $(0 < t < \infty)$) is a lower solution of (1.1) (or (B1)) on $0 < t < \infty$.

The main result for the application in Theorem 8 is:

Proposition 2B. Let the assumptions (B3) be satisfied. Let ϕ be the solution, let w be an upper solution, and let v be a lower solution of (B1) on $0 < t < \gamma$. Then

$$\text{(B9)} \qquad v(t) < \phi(t) < w(t) \qquad (0 < t < \gamma) .$$

In Theorem 8 one takes $v(t) = \phi_0(t)$, when ϕ_0 is the solution of (1.6), and one shows that w given by (10.1) is an upper solution.

Proof of Proposition 2B. We shall prove the second inequality in (B9); the first is proved in a similar way. If $w(0) > g(0)$ the result follows directly from Proposition 1B with v replaced by ϕ, D_-v by ϕ', and D_-w by $w'_+(t)$ (and one uses Lemma 1.2.2 of [6]).

If $w(0) = g(0)$ one has from (B1)

$$-\mu\phi'(0^+) = f_a(g(0))(0) = \int_{-\infty}^{0} a(-s)F(g(0),g(s))ds$$

and from the definition of upper solution

$$-\mu w'_+(0^+) < f_a(w(0))(0) = \int_{-\infty}^{0} a(-s)F(g(0),g(s))ds \ .$$

Therefore

$$w'_+(0^+) > \phi'(0^+) \ ,$$

and there exists $\varepsilon > 0$, such that the function m defined by $m(t) = w(t) - \phi(t)$ is strictly increasing on $0 < t \leq \varepsilon$. Thus $w(\varepsilon) > \phi(\varepsilon)$, and applying Proposition 1B as above on the interval $[\varepsilon, \gamma)$ shows that for any $\varepsilon > 0$, $\phi(t) < w(t)$ $(\varepsilon \leq t < \gamma)$. This completes the proof of Proposition 2B.

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We study the initial value problem for the nonlinear Volterra integrodifferential equation

$$(+) \quad \begin{cases} -\mu y'(t) = \int_{-\infty}^{t} a(t-s)F(y(t),y(s))ds & (t > 0) \\ y(t) = g(t) & (-\infty < t \le 0), \end{cases}$$

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20. ABSTRACT (Continued)

where $\mu > 0$ is a small parameter, a is a given real kernel, and F, g are given real functions; (+) models the elongation ratio of a homogeneous filament of a certain polyethylene which is stretched on the time interval $(-\infty, 0]$, then released and allowed to undergo elastic recovery for $t > 0$. Under assumptions which include physically interesting cases of the given functions a, F, g, we discuss qualitative properties of the solution of (+) and of the corresponding reduced problem when $\mu = 0$, and the relation between them as $\mu \to 0^+$, both for t near zero (where a boundary layer occurs) and for large t. In particular, we show that in general the filament does not recover its original length, and that the Newtonian term $-\mu y'$ in (+) has little effect on the ultimate recovery but significant effect during the early part of the recovery.